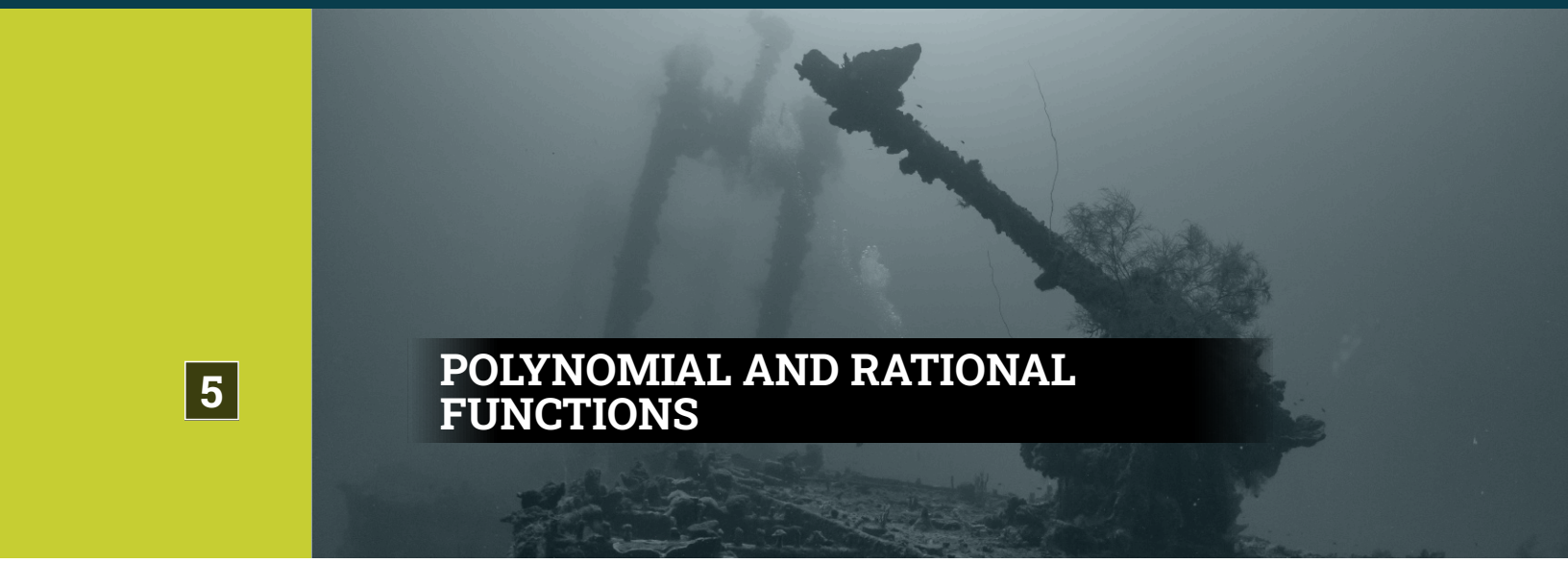


5 POLYNOMIAL AND RATIONAL FUNCTIONS

Whether they think about it in mathematical terms or not, scuba divers must consider the impact of functional relationships in order to remain safe. The gas laws, which are a series of relations and equations that describe the behavior of most gases, play a core role in diving. This diver, near the wreck of a World War II Japanese ocean liner turned troop transport, must remain attentive to gas laws during their dive and as they ascend to the surface. (credit: "Aikoku - Aft Gun": modification of work by montereydiver/flickr)

Chapter Outline

5.1 Quadratic Functions
5.2 Power Functions and Polynomial Functions
5.3 Graphs of Polynomial Functions
5.4 Dividing Polynomials
5.5 Zeros of Polynomial Functions
5.6 Rational Functions
5.7 Inverses and Radical Functions
5.8 Modeling Using Variation

Introduction to Polynomial and Rational Functions

You don't need to dive very deep to feel the effects of pressure. As a person in their neighborhood pool moves eight, ten, twelve feet down, they often feel pain in their ears as a result of water and air pressure differentials. Pressure plays a much greater role at ocean diving depths.

Scuba and free divers are constantly negotiating the effects of pressure in order to experience enjoyable, safe, and productive dives. Gases in a person's respiratory system and diving apparatus interact according to certain physical properties, which upon discovery and evaluation are collectively known as the gas laws. Some are conceptually simple, such as the inverse relationship regarding pressure and volume, and others are more complex. While their formulas seem more straightforward than many you will encounter in this chapter, the gas laws are generally polynomial expressions.

5.1 Quadratic Functions

Learning Objectives

In this section, you will:

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function’s minimum or maximum value.
- Solve problems involving a quadratic function’s minimum or maximum value.



Figure 1 An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

Curved antennas, such as the ones shown in Figure 1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 2.

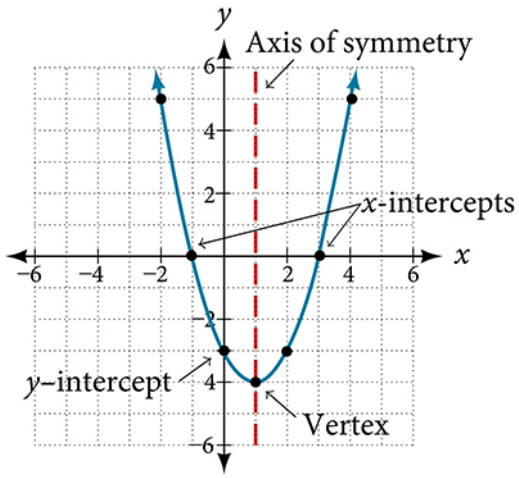


Figure 2

The *y*-intercept is the point at which the parabola crosses the *y*-axis. The *x*-intercepts are the points at which the parabola crosses the *x*-axis. If they exist, the *x*-intercepts represent the **zeros,** or **roots**, of the quadratic function, the values of x at which $y = 0$.

EXAMPLE 1

Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y- intercept of the parabola shown in Figure 3.

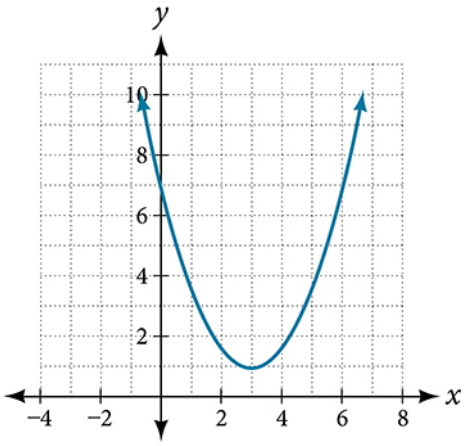


Figure 3

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x- axis, so it has no zeros. It crosses the y- axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x- intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1, b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x- intercepts, those points where the parabola crosses the x- axis, occur at $(-3, 0)$ and $(-1, 0)$.

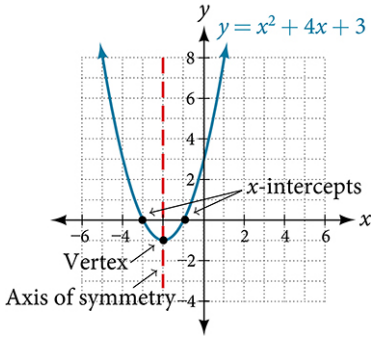


Figure 4

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k$$

where $(h,\ k)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2,\ 4)$.

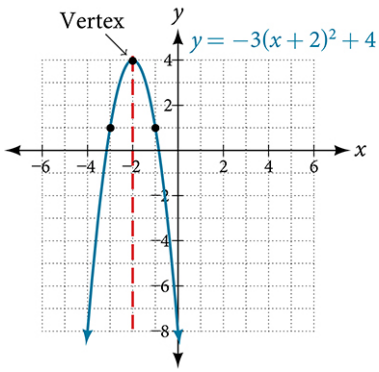


Figure 5

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 6 is the graph of this basic function.

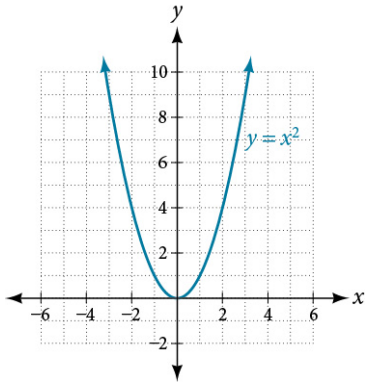


Figure 6

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 5, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x- value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x- value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 5, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2 + k &= ax^2 + bx + c \\ ax^2 - 2ahx + (ah^2 + k) &= ax^2 + bx + c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah = b, \text{ so } h = -\frac{b}{2a}$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2 + k &= c \\ k &= c - ah^2 \\ &= c - a - \left(\frac{b}{2a}\right)^2 \\ &= c - \frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h) = k$.

Forms of Quadratic Functions

A quadratic function is a polynomial function of degree two. The graph of a quadratic function is a parabola.

The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$.

The **standard form of a quadratic function** is $f(x) = a(x - h)^2 + k$ where $a \neq 0$.

The vertex (h, k) is located at

$$h = -\frac{b}{2a}, \quad k = f(h) = f\left(\frac{-b}{2a}\right)$$



HOW TO

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x) = a(x-h)^2 + k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. Expand and simplify to write in general form.

EXAMPLE 2

Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 7 as a transformation of $f(x) = x^2$, and then expand the formula, and simplify terms to write the equation in general form.

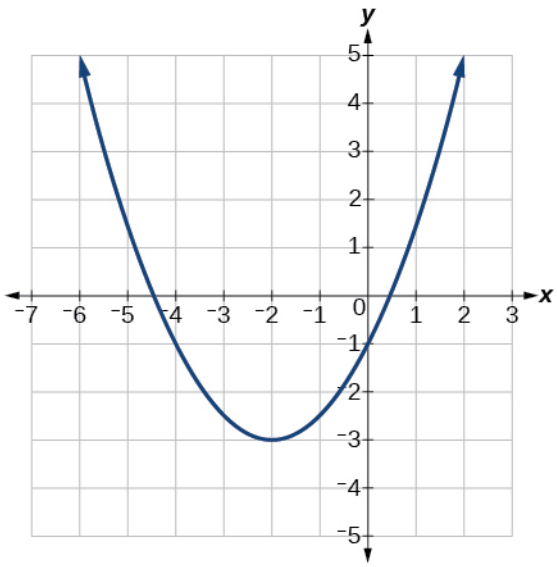


Figure 7

Solution

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x-(-2))^2 - 3 = a(x+2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$\begin{aligned} -1 &= a(0+2)^2 - 3 \\ 2 &= 4a \\ a &= \tfrac{1}{2} \end{aligned}$$

In standard form, the algebraic model for this graph is $(g)x = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} g(x) &= \tfrac{1}{2}(x+2)^2 - 3 \\ &= \tfrac{1}{2}(x+2)(x+2) - 3 \\ &= \tfrac{1}{2}(x^2 + 4x + 4) - 3 \\ &= \tfrac{1}{2}x^2 + 2x + 2 - 3 \\ &= \tfrac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(x+2)^2 - 3$. Next, select TBLSET, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select TABLE. See Table 1.

x	–6	–4	–2	0	2
y	5	–1	–3	–1	5

Table 1

The ordered pairs in the table correspond to points on the graph.

TRY IT #1 A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 8.

Assume that the point (–4, 7) is the highest point of the basketball's trajectory. Find an equation for the path of the ball. Does the shooter make the basket?

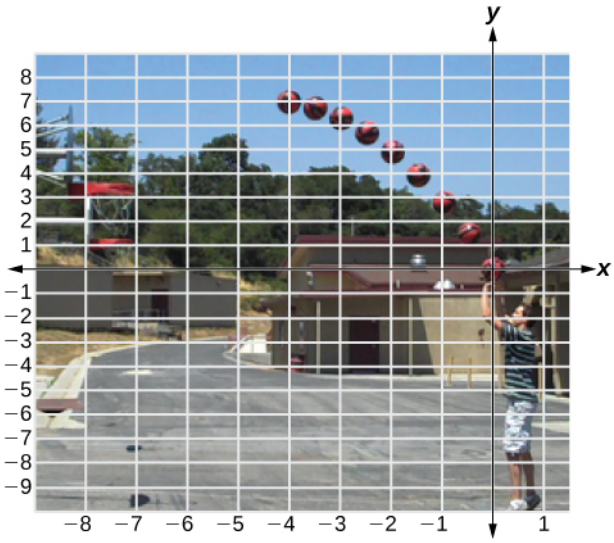


Figure 8 (credit: modification of work by Dan Meyer)

HOW TO

Given a quadratic function in general form, find the vertex of the parabola.

1. Identify $a,\ b,$ and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

EXAMPLE 3

Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(2)} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

The vertical coordinate of the vertex will be at

$$\begin{aligned} k &= f(h) \\ &= f\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \\ &= \frac{5}{2} \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. First, find the horizontal coordinate of the vertex. Then find the vertical coordinate of the vertex. Substitute the values into standard form, using the " a " from the general form.

$$\begin{aligned} f(x) &= ax^2+bx+c \\ f(x) &= 2x^2-6x+7 \end{aligned}$$

The standard form of a quadratic function prior to writing the function then becomes the following:

$$f(x)=2\left(x-\frac{3}{2}\right)^2+\frac{5}{2}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, k, and where it occurs, x.

TRY IT #2 Given the equation $g(x)=13+x^2-6x$, write the equation in general form and then in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all *y*-values greater than or equal to the *y*-coordinate at the turning point or less than or equal to the *y*-coordinate at the turning point, depending on whether the parabola opens up or down.

Domain and Range of a Quadratic Function

The domain of any quadratic function is all real numbers unless the context of the function presents some restrictions.

The range of a quadratic function written in general form $f(x)=ax^2+bx+c$ with a positive a value is $f(x)\geq f\left(-\frac{b}{2a}\right)$, or $\left[f\left(-\frac{b}{2a}\right),\infty\right)$; the range of a quadratic function written in general form with a negative a value is $f(x)\leq f\left(-\frac{b}{2a}\right)$, or $\left(-\infty,f\left(-\frac{b}{2a}\right)\right]$.

The range of a quadratic function written in standard form $f(x)=a(x-h)^2+k$ with a positive a value is $f(x)\geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x)\leq k$.

HOW TO

Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x)\geq k$, or $\left[k,\infty\right)$. If the parabola has a maximum, the range is given by $f(x)\leq k$, or $\left(-\infty,k\right]$.

EXAMPLE 4

Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x)=-5x^2+9x-1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x- value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{9}{2(-5)} \\ &= \frac{9}{10} \end{aligned}$$

The maximum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{9}{10}\right) &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\ &= \frac{61}{20} \end{aligned}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

TRY IT #3 Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in Figure 9.

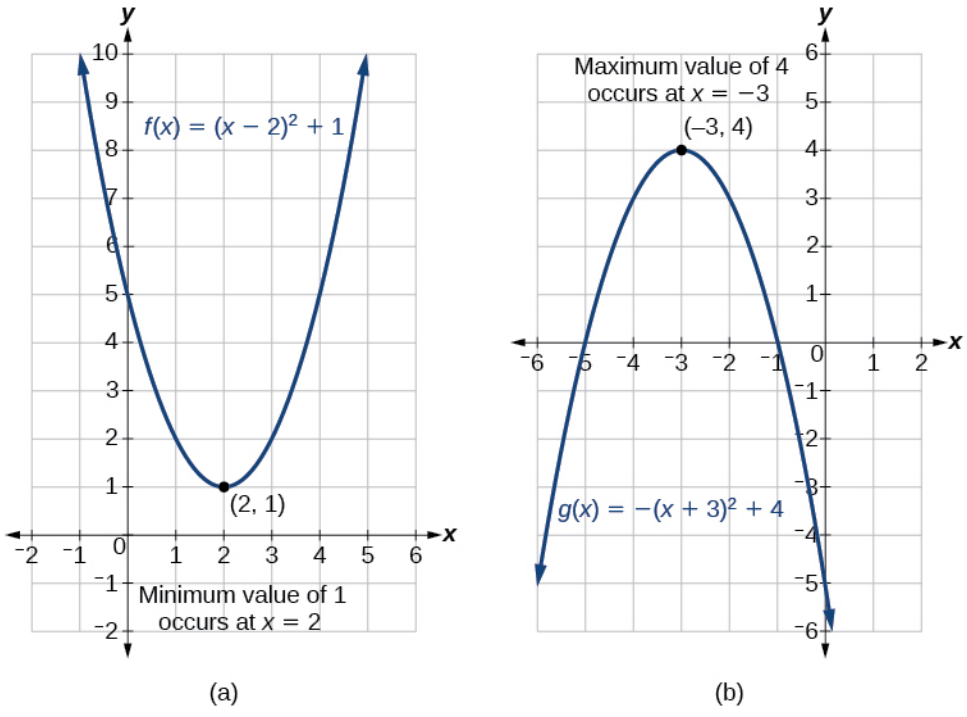


Figure 9

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

EXAMPLE 5

Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

(a) Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

(b) What dimensions should she make her garden to maximize the enclosed area?

Solution

Let's use a diagram such as Figure 10 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

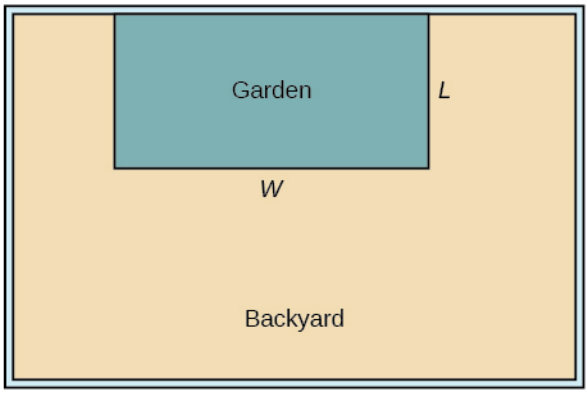


Figure 10

(a) We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L.

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$\begin{aligned} A &= LW = L(80 - 2L) \\ A(L) &= 80L - 2L^2 \end{aligned}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L) = -2L^2 + 80L.$$

(b) The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{80}{2(-2)} \\ &= 20 \end{aligned} \quad \text{and} \quad \begin{aligned} k &= A(20) \\ &= 80(20) - 2(20)^2 \\ &= 800 \end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 11.

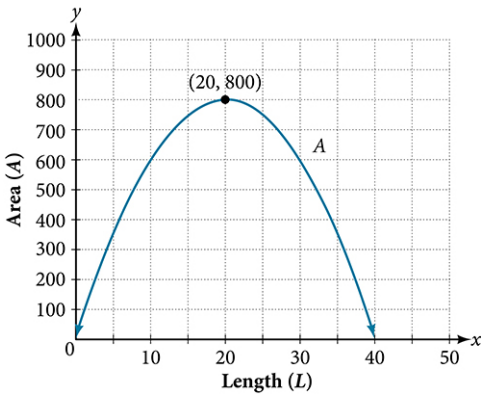


Figure 11

HOW TO

Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for a revenue function.
2. Find the vertex of the quadratic equation.
3. Determine the *y*-value of the vertex.

EXAMPLE 6

Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84{,}000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79{,}000$. From this we can find a linear equation relating the two quantities. The slope will be

$$\begin{aligned} m &= \frac{79{,}000-84{,}000}{32-30} \\ &= \frac{-5{,}000}{2} \\ &= -2{,}500 \end{aligned}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the *y*-intercept.

$$\begin{aligned} Q &= -2500p + b && \text{Substitute in the point } Q = 84{,}000 \text{ and } p = 30 \\ 84{,}000 &= -2500(30) + b && \text{Solve for } b \\ b &= 159{,}000 \end{aligned}$$

This gives us the linear equation $Q = -2{,}500p + 159{,}000$ relating cost and subscribers. We now return to our revenue equation.

$$\begin{aligned} \text{Revenue} &= pQ \\ \text{Revenue} &= p(-2{,}500p + 159{,}000) \\ \text{Revenue} &= -2{,}500p^2 + 159{,}000p \end{aligned}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$\begin{aligned} h &= -\frac{159{,}000}{2(-2{,}500)} \\ &= 31.8 \end{aligned}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\begin{aligned} \text{maximum revenue} &= -2{,}500(31.8)^2 + 159{,}000(31.8) \\ &= 2{,}528{,}100 \end{aligned}$$

Analysis

This could also be solved by graphing the quadratic as in Figure 12. We can see the maximum revenue on a graph of the quadratic function.

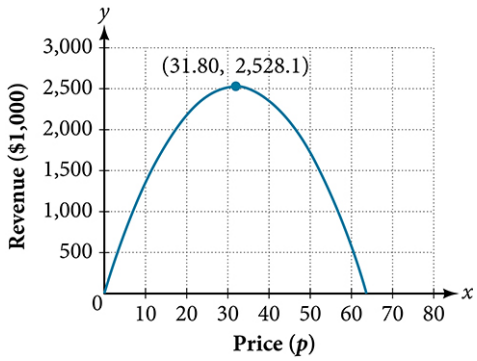


Figure 12

Finding the *x*- and *y*-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y- intercept of a quadratic by evaluating the function at an input of zero, and we find the x- intercepts at locations where the output is zero. Notice in Figure 13 that the number of x- intercepts can vary depending upon the location of the graph.

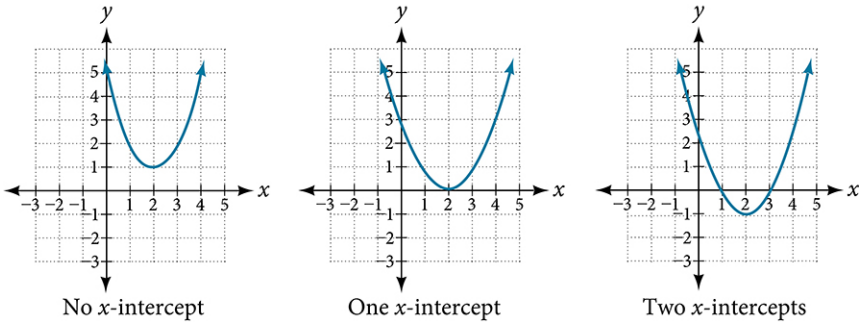


Figure 13 Number of x-intercepts of a parabola

HOW TO

Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.

2. Solve the quadratic equation $f(x) = 0$ to find the *x*-intercepts.

EXAMPLE 7

Finding the *y*- and *x*-Intercepts of a Parabola

Find the *y*- and *x*-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

We find the *y*-intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 3(0)^2 + 5(0) - 2 \\ &= -2 \end{aligned}$$

So the *y*-intercept is at $(0, -2)$.

For the *x*-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2)$$

So the *x*-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the *y*-axis at $(0, -2)$. We can also confirm that the graph crosses the *x*-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 14

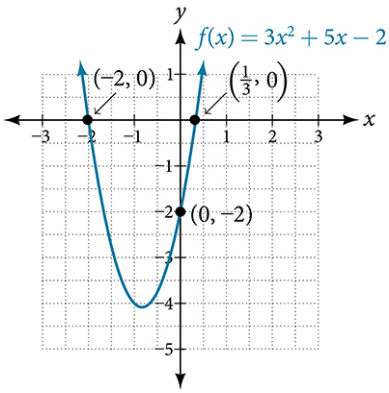


Figure 14

Rewriting Quadratics in Standard Form

In Example 7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

HOW TO

Given a quadratic function, find the x- intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x- intercepts.

EXAMPLE 8

Finding the *x*-Intercepts of a Parabola

Find the x- intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\tfrac{b}{2a} & \qquad k &= f(-1) \\ &= -\tfrac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

We can check our work by graphing the given function on a graphing utility and observing the x- intercepts. See Figure 15.

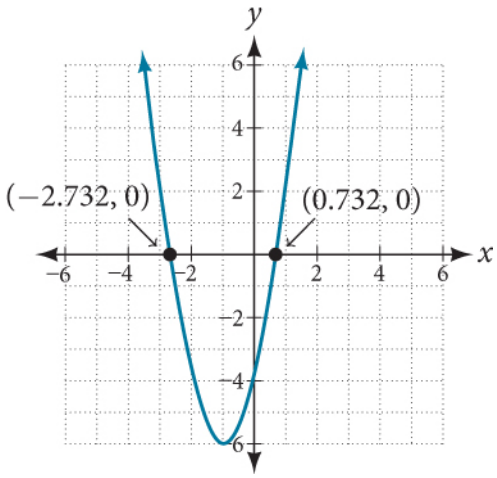


Figure 15

Analysis

We could have achieved the same results using the quadratic formula. Identify $a = 2, b = 4$ and $c = -4$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2-4(2)(-4)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{48}}{4} \\ &= \frac{-4 \pm \sqrt{3(16)}}{4} \\ &= -1 \pm \sqrt{3} \end{aligned}$$

So the *x*-intercepts occur at $\left(-1-\sqrt{3}, 0\right)$ and $\left(-1+\sqrt{3}, 0\right)$.

TRY IT #4 In a Try It, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the *y*- and *x*-intercepts (if any).

EXAMPLE 9

Applying the Vertex and *x*-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

(a) When does the ball reach the maximum height? (b) What is the maximum height of the ball?

(c) When does the ball hit the ground?

Solution

(a) The ball reaches the maximum height at the vertex of the parabola.

$$\begin{aligned} h &= -\frac{80}{2(-16)} \\ &= \frac{80}{32} \\ &= \frac{5}{2} \\ &= 2.5 \end{aligned}$$

The ball reaches a maximum height after 2.5 seconds.

(b) To find the maximum height, find the *y*- coordinate of the vertex of the parabola.

$$\begin{aligned} k &= H\left(-\frac{b}{2a}\right) \\ &= H(2.5) \\ &= -16(2.5)^2 + 80(2.5) + 40 \\ &= 140 \end{aligned}$$

The ball reaches a maximum height of 140 feet.

ⓒ To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.
We use the quadratic formula.

$$\begin{aligned} t &= \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \\ &= \frac{-80 \pm \sqrt{8960}}{-32} \end{aligned}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 16.

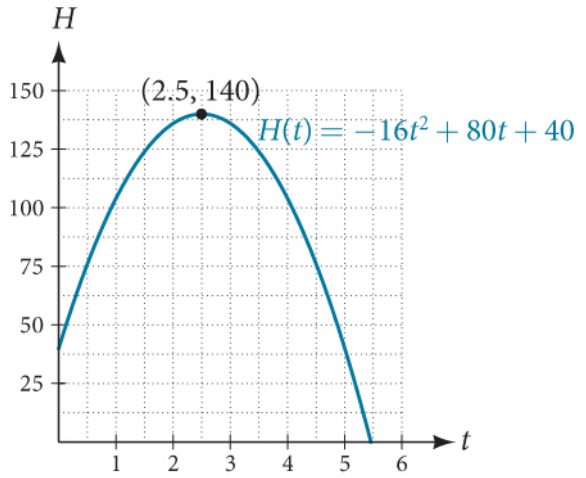


Figure 16

Note that the graph does not represent the physical path of the ball upward and downward. Keep the quantities on each axis in mind while interpreting the graph.

TRY IT #5 A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

ⓐ When does the rock reach the maximum height?
ⓑ What is the maximum height of the rock? ⓒ When does the rock hit the ocean?

MEDIA

Access these online resources for additional instruction and practice with quadratic equations.

Graphing Quadratic Functions in General Form (https://openstax.org/l/graphquadgen)
Graphing Quadratic Functions in Standard Form (https://openstax.org/l/graphquadstan)
Quadratic Function Review (https://openstax.org/l/quadfuncrev)
Characteristics of a Quadratic Function (https://openstax.org/l/characterquad)

5.1 SECTION EXERCISES

Verbal

1. Explain the advantage of writing a quadratic function in standard form.
2. How can the vertex of a parabola be used in solving real-world problems?
3. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.
4. What is another name for the standard form of a quadratic function?
5. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Algebraic

For the following exercises, rewrite the quadratic functions in vertex form and give the vertex.

6. $f(x) = x^2 - 12x + 32$
7. $g(x) = x^2 + 2x - 3$
8. $f(x) = x^2 - x$
9. $f(x) = x^2 + 5x - 2$
10. $h(x) = 2x^2 + 8x - 10$
11. $k(x) = 3x^2 - 6x - 9$
12. $f(x) = 2x^2 - 6x$
13. $f(x) = 3x^2 - 5x - 1$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

14. $y(x) = 2x^2 + 10x + 12$
15. $f(x) = 2x^2 - 10x + 4$
16. $f(x) = -x^2 + 4x + 3$
17. $f(x) = 4x^2 + x - 1$
18. $h(t) = -4t^2 + 6t - 1$
19. $f(x) = \frac{1}{2}x^2 + 3x + 1$
20. $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

21. $f(x) = (x-3)^2 + 2$
22. $f(x) = -2(x+3)^2 - 6$
23. $f(x) = x^2 + 6x + 4$
24. $f(x) = 2x^2 - 4x + 2$
25. $k(x) = 3x^2 - 6x - 9$

For the following exercises, use the vertex (h, k) and a point on the graph (x, y) to find the general form of the equation of the quadratic function.

26. $(h, k) = (2, 0), (x, y) = (4, 4)$
27. $(h, k) = (-2, -1), (x, y) = (-4, 3)$
28. $(h, k) = (0, 1), (x, y) = (2, 5)$
29. $(h, k) = (2, 3), (x, y) = (5, 12)$
30. $(h, k) = (-5, 3), (x, y) = (2, 9)$
31. $(h, k) = (3, 2), (x, y) = (10, 1)$
32. $(h, k) = (0, 1), (x, y) = (1, 0)$
33. $(h, k) = (1, 0), (x, y) = (0, 1)$

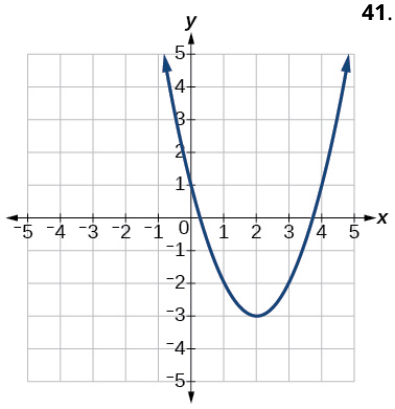
Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

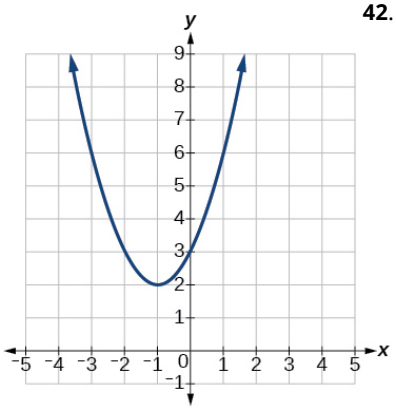
34. $f(x) = x^2 - 2x$
35. $f(x) = x^2 - 6x - 1$
36. $f(x) = x^2 - 5x - 6$
37. $f(x) = x^2 - 7x + 3$
38. $f(x) = -2x^2 + 5x - 8$
39. $f(x) = 4x^2 - 12x - 3$

For the following exercises, write the equation for the graphed quadratic function.

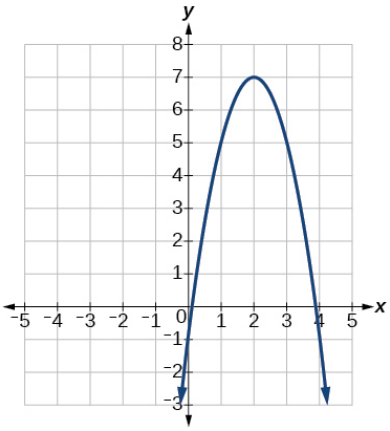
40.



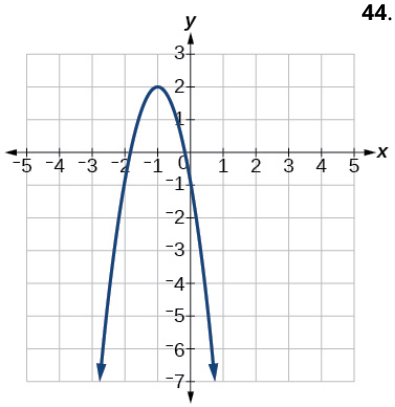
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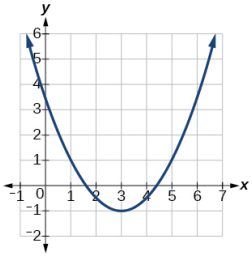
42.



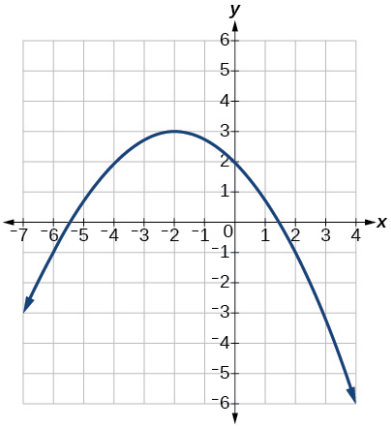
43.



44.



45.



Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

46.

x	−2	−1	0	1	2
y	5	2	1	2	5

47.

x	−2	−1	0	1	2
y	1	0	1	4	9

48.

x	−2	−1	0	1	2
y	−2	1	2	1	−2

49.

x	−2	−1	0	1	2
y	−8	−3	0	1	0

50.

x	−2	−1	0	1	2
y	8	2	0	2	8

Technology

For the following exercises, use a calculator to find the answer.

51. Graph on the same set of axes the functions $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?

52. Graph on the same set of axes $f(x) = x^2$, $f(x) = x^2 + 2$ and $f(x) = x^2$, $f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

53. Graph on the same set of axes $f(x) = x^2$, $f(x) = (x-2)^2$, $f(x-3)^2$, and $f(x) = (x+4)^2$. What appears to be the effect of adding or subtracting those numbers?

54. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = \frac{-32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

55. A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

56. Vertex $(1, -2)$, opens up.

57. Vertex $(-1, 2)$ opens down.

58. Vertex $(-5, 11)$, opens down.

59. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

60. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the y- axis.

61. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y- axis.

62. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y- axis.

63. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the y- axis.

64. Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y- axis.

65. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

66. Find the dimensions of the rectangular dog park producing the greatest enclosed area given 200 feet of fencing.

67. Find the dimensions of the rectangular dog park split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

68. Find the dimensions of the rectangular dog park producing the greatest enclosed area split into 3 sections of the same size given 500 feet of fencing.

69. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

70. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

71. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

72. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

73. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

74. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

75. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

5.2 Power Functions and Polynomial Functions

Learning Objectives

In this section, you will:

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.



Figure 1 (credit: Jason Bay, Flickr)

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 1.

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

Table 1

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

Before we can understand the bird problem, it will be helpful to understand a different type of function. A **power function** is a function with a single term that is the product of a real number, a **coefficient,** and a variable raised to a fixed real number.

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Power Function

A **power function** is a function that can be represented in the form

$$f(x) = kx^p$$

where k and p are real numbers, and k is known as the **coefficient**.

Q&A **Is $f(x) = 2^x$ a power function?**

No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an exponential function, not a power function.

EXAMPLE 1

Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identity function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{\frac{1}{2}}$ or $f(x) = x^{\frac{1}{3}}$.

TRY IT #1 Which functions are power functions?

$$f(x) = 2x \cdot 4x^3$$
$$g(x) = -x^5 + 5x^3$$
$$h(x) = \frac{2x^5 - 1}{3x^2 + 4}$$

Identifying End Behavior of Power Functions

Figure 2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and $h(x) = x^6$, which are all power functions with even, positive integer powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

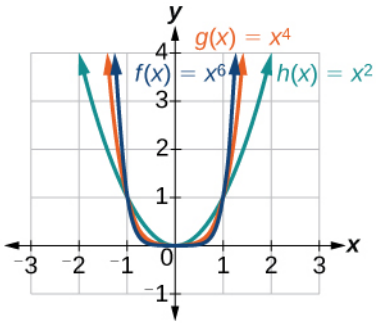


Figure 2 Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically

written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the positive even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

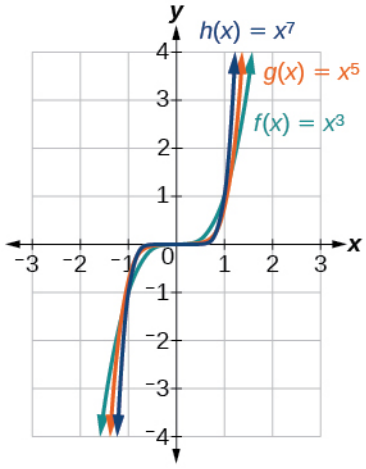


Figure 3 Odd-power functions

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 2 we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y- axis. In Figure 3 we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Figure 4 shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

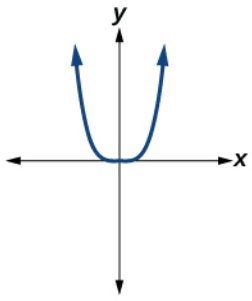
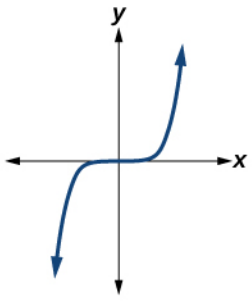
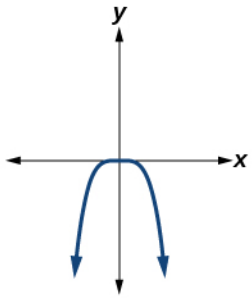
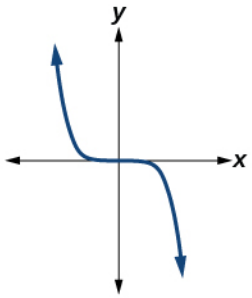
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to \infty$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to -\infty$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to -\infty$

Figure 4

HOW TO

Given a power function $f(x) = kx^n$ where n is a positive integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure 4 to identify the end behavior.

EXAMPLE 2

Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure 5.

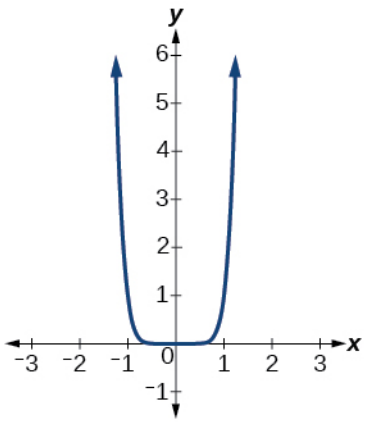


Figure 5

EXAMPLE 3

Identifying the End Behavior of a Power Function.
Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x- axis of the graph of $f(x) = x^9$. Figure 6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$

$$\text{as } x \to \infty,\ f(x) \to -\infty$$

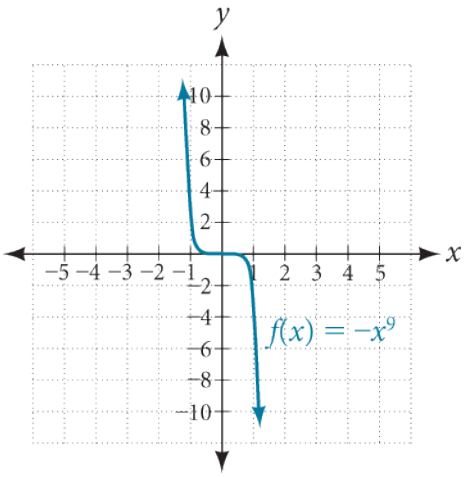


Figure 6

Analysis

We can check our work by using the table feature on a graphing utility.

x	$f(x)$
–10	1,000,000,000
–5	1,953,125
0	0
5	–1,953,125
10	–1,000,000,000

Table 2

We can see from Table 2 that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

TRY IT #2 Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a **polynomial function**. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Polynomial Functions

Let n be a non-negative integer. A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number, but $a_n \neq 0$. Each expression $a_i x^i$ is a **term of a polynomial function**.

EXAMPLE 4

Identifying Polynomial Functions

Which of the following are polynomial functions?

$$\begin{aligned} f(x) &= 2x^3 \cdot 3x + 4 \\ g(x) &= -x(x^2 - 4) \\ h(x) &= 5\sqrt{x+2} \end{aligned}$$

Solution

The first two functions are examples of polynomial functions because they can be written in the form $f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$, where the powers are non-negative integers and the coefficients are real numbers.

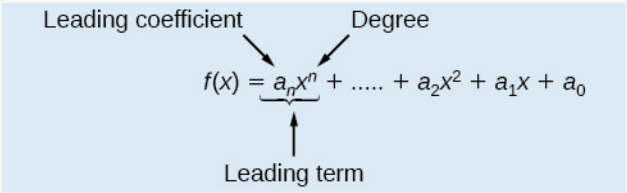
- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.



When a polynomial is written in this way, we say that it is in general form.

HOW TO

Given a polynomial function, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree of the function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

EXAMPLE 5

Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$\begin{aligned} f(x) &= 3+2x^2-4x^3 \\ g(t) &= 5t^5-2t^3+7t \\ h(p) &= 6p-p^3-2 \end{aligned}$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$. The leading coefficient is the coefficient of that term, -1.

TRY IT #3 Identify the degree, leading term, and leading coefficient of the polynomial $f(x)=4x^2-x^6+2x-6$.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the power function consisting of the leading term. See Table 3.

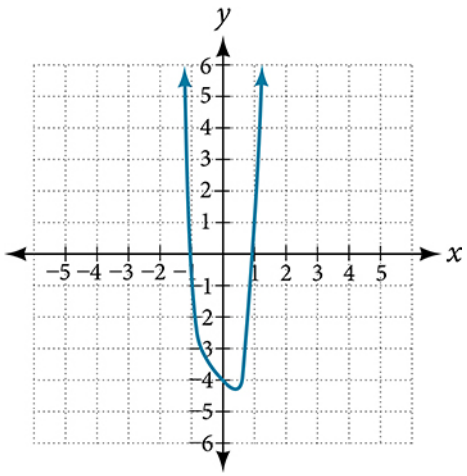
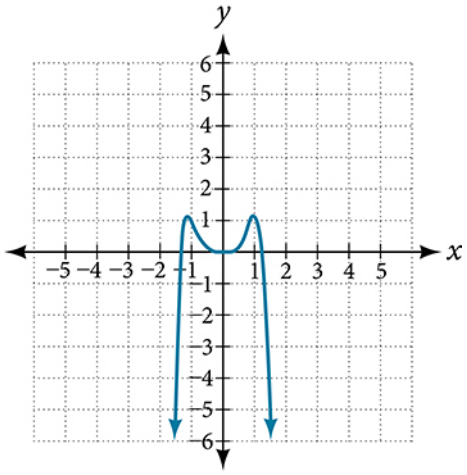
Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	

Table 3

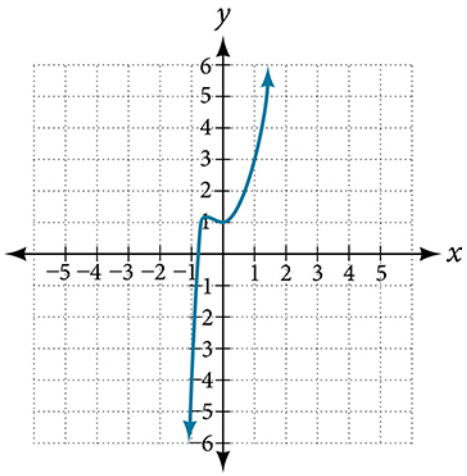
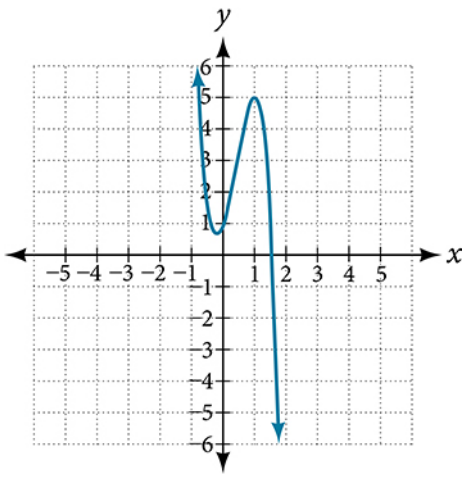
Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Table 3

EXAMPLE 6

Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 7.

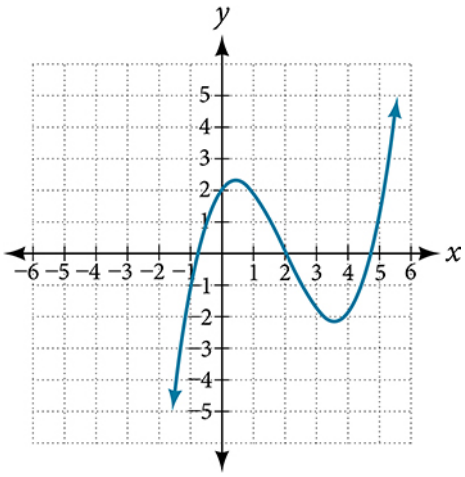


Figure 7

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to \infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

TRY IT #4 Describe the end behavior, and determine a possible degree of the polynomial function in Figure 8.

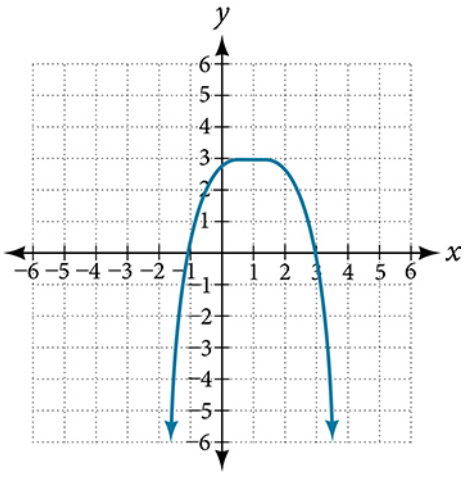


Figure 8

EXAMPLE 7

Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the

leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2\left(x^2+3x-4\right) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to -\infty$$

TRY IT #5 Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the *y*-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one *y*-intercept $(0, a_0)$. The *x*-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one *x*-intercept. See Figure 9.

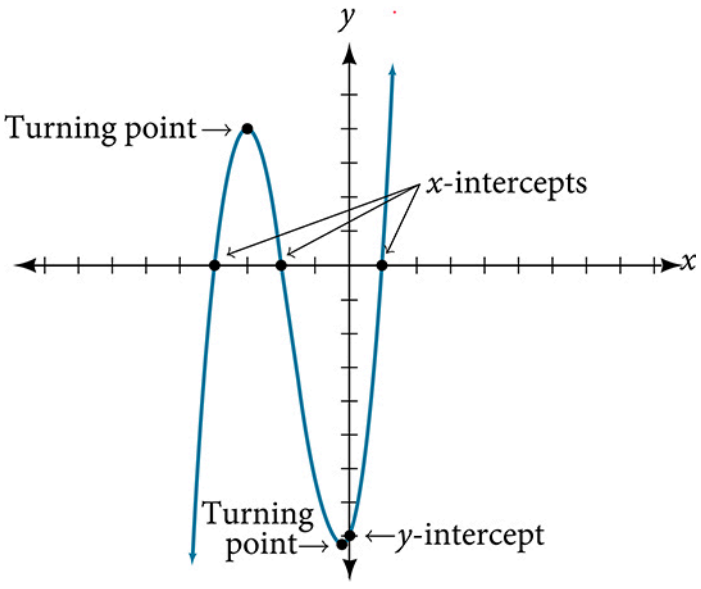


Figure 9

Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The *y*-intercept is the point at which the function has an input value of zero. The *x*-intercepts are the points at which the output value is zero.

HOW TO

Given a polynomial function, determine the intercepts.

1. Determine the *y*-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the *x*-intercepts by solving for the input values that yield an output value of zero.

EXAMPLE 8

Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the *y*- and *x*-intercepts.

Solution

The *y*-intercept occurs when the input is zero so substitute 0 for x.

$$\begin{aligned} f(0) &= f(0) = (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The *y*-intercept is (0, 8).

The *x*-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{aligned} x-2 &= 0 \quad \text{or} \quad x+1 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 2 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 4 \end{aligned}$$

The *x*-intercepts are $(2, 0), (-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in Figure 10.

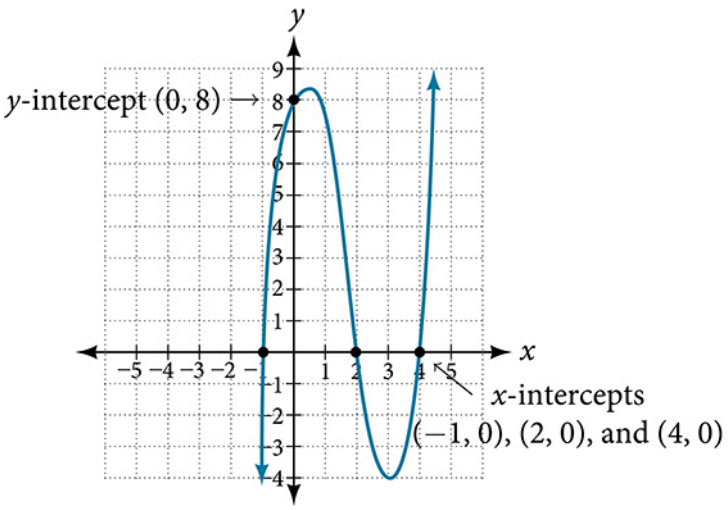


Figure 10

EXAMPLE 9

Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the *y*- and *x*-intercepts.

Solution

The *y*-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The *y*-intercept is $(0, -45)$.

The *x*-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= \left(x^2 - 9\right)\left(x^2 + 5\right) \\ &= (x-3)(x+3)\left(x^2 + 5\right) \end{aligned}$$

$$0 = (x-3)(x+3)\left(x^2 + 5\right)$$

$$\begin{array}{lllllll} x-3 = 0 & \text{or} & x+3 = 0 & \text{or} & x^2+5 = 0 \\ x = 3 & \text{or} & x = -3 & \text{or} & \text{(no real solution)} \end{array}$$

The *x*-intercepts are $(3, 0)$ and $(-3, 0)$.

We can see these intercepts on the graph of the function shown in Figure 11. We can see that the function is even because $f(x) = f(-x)$.

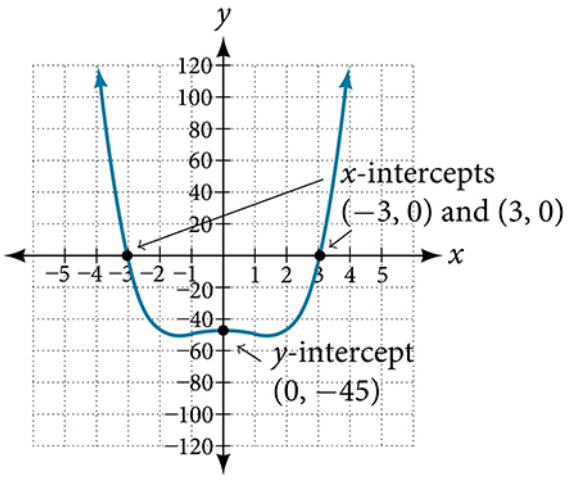


Figure 11

TRY IT #6 Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the *y*- and *x*-intercepts.

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of *x*-intercepts and the number of turning points. A polynomial function of *n*th degree is the product of *n* factors, so it will have at most *n* roots or zeros, or *x*-intercepts. The graph of the polynomial function of degree *n* must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree *n* will have, at most, *n* *x*-intercepts and $n - 1$ turning points.

EXAMPLE 10

Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution

The polynomial has a degree of 10, so there are at most 10 x-intercepts and at most 9 turning points.

TRY IT #7 Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$.

EXAMPLE 11

Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 12 based on its intercepts and turning points?

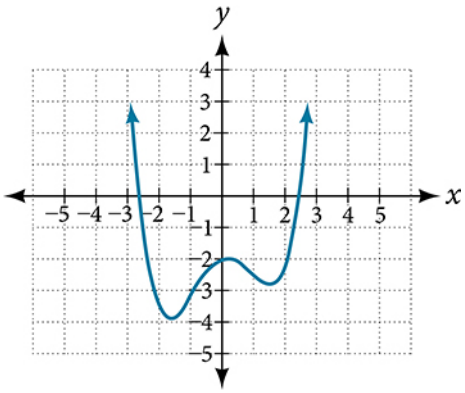


Figure 12

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 13.

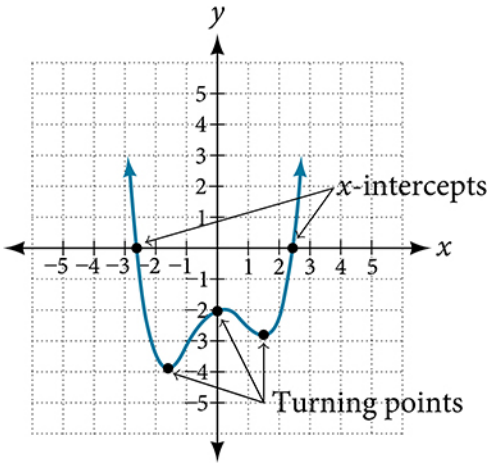


Figure 13

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

TRY IT #8 What can we conclude about the polynomial represented by the graph shown in Figure 14 based on its intercepts and turning points?

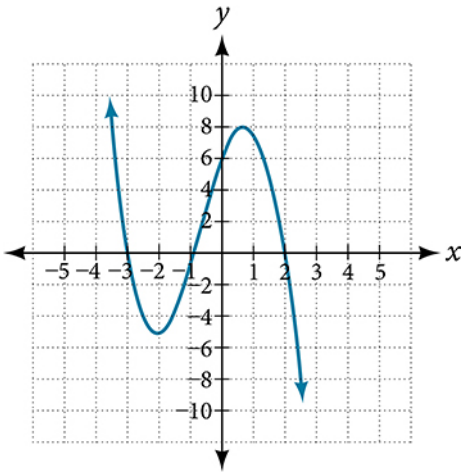


Figure 14

EXAMPLE 12

Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution

The *y*-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The *y*-intercept is $(0, 0)$.

The *x*-intercepts are found by determining the zeros of the function.

$$0 = -4x(x+3)(x-4)$$

$$\begin{aligned} x &= 0 \quad \text{or} \quad x+3 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 0 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 4 \end{aligned}$$

The *x*-intercepts are $(0, 0)$, $(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

TRY IT #9 Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

MEDIA

Access these online resources for additional instruction and practice with power and polynomial functions.

Find Key Information about a Given Polynomial Function (https://openstax.org/l/keyinfopoly)
End Behavior of a Polynomial Function (https://openstax.org/l/endbehavior)
Turning Points and *x*- intercepts of Polynomial Functions (https://openstax.org/l/turningpoints)
Least Possible Degree of a Polynomial Function (https://openstax.org/l/leastposdegree)

5.2 SECTION EXERCISES

Verbal

1. Explain the difference between the coefficient of a power function and its degree.

2. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3. In general, explain the end behavior of a polynomial with odd degree if the leading coefficient is positive.

4. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

5. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior?
As $x \to -\infty$, $f(x) \to -\infty$
and as
$x \to \infty$, $f(x) \to -\infty$.

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

6. $f(x) = x^5$

7. $f(x) = \left(x^2\right)^3$

8. $f(x) = x - x^4$

9. $f(x) = \frac{x^2}{x^2-1}$

10. $f(x) = 2x\,(x+2)\,(x-1)^2$

11. $f(x) = 3^{x+1}$

For the following exercises, find the degree and leading coefficient for the given polynomial.

12. $-3x^4$

13. $7 - 2x^2$

14. $-2x^2 - 3x^5 + x - 6$

15. $x\left(4 - x^2\right)(2x+1)$

16. $x^2(2x-3)^2$

For the following exercises, determine the end behavior of the functions.

17. $f\,(x) = x^4$

18. $f\,(x) = x^3$

19. $f\,(x) = -x^4$

20. $f\,(x) = -x^9$

21. $f(x) = -2x^4 - 3x^2 + x - 1$

22. $f(x) = 3x^2 + x - 2$

23. $f(x) = x^2\left(2x^3 - x + 1\right)$

24. $f(x) = (2-x)^7$

For the following exercises, find the intercepts of the functions.

25. $f\,(t) = 2\,(t-1)\,(t+2)\,(t-3)$

26. $g\,(n) = -2\,(3n-1)\,(2n+1)$

27. $f(x) = x^4 - 16$

28. $f(x) = x^3 + 27$

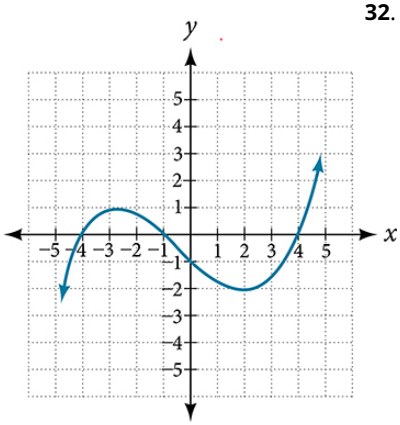
29. $f(x) = x\left(x^2 - 2x - 8\right)$

30. $f(x) = (x+3)\left(4x^2 - 1\right)$

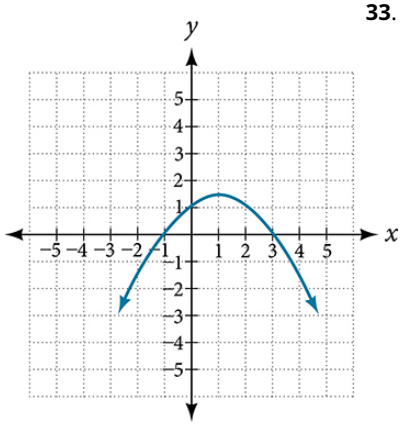
Graphical

For the following exercises, determine the least possible degree of the polynomial function shown.

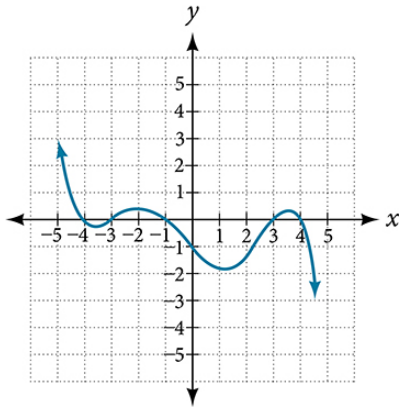
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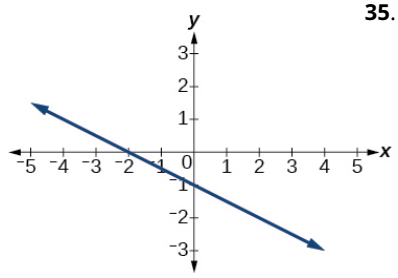
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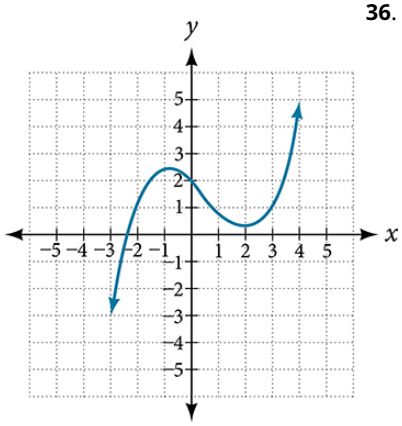
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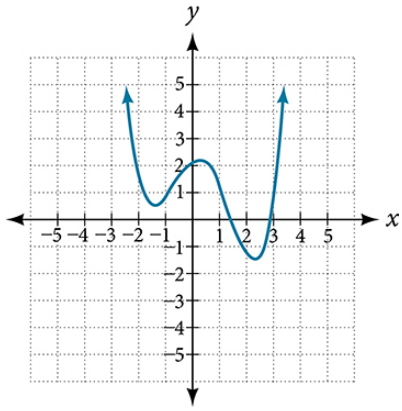
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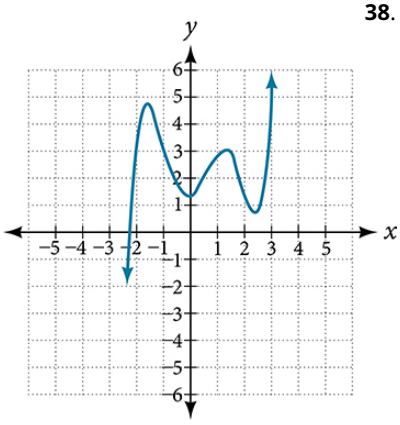
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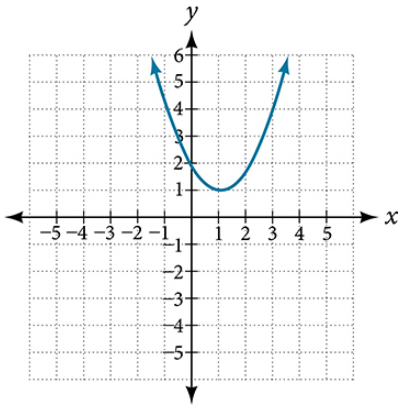
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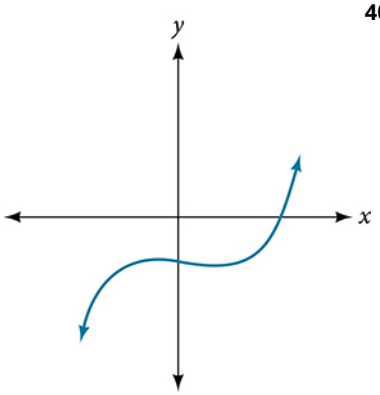


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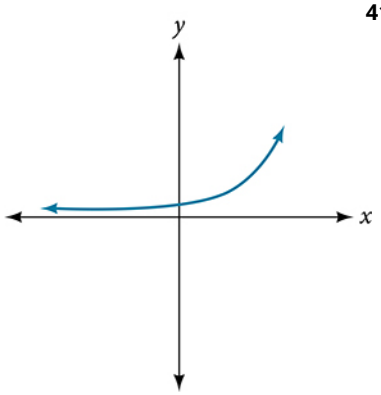


For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

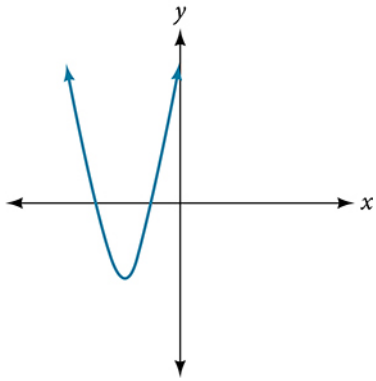
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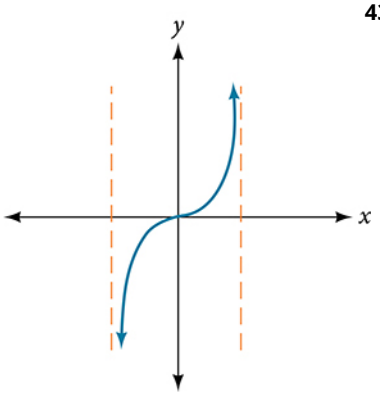
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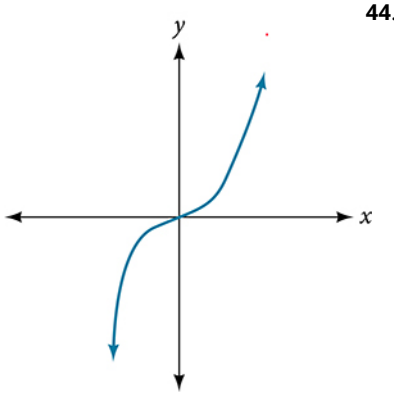
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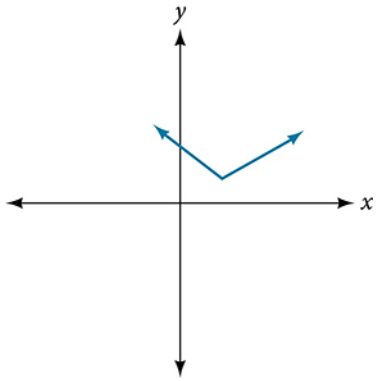
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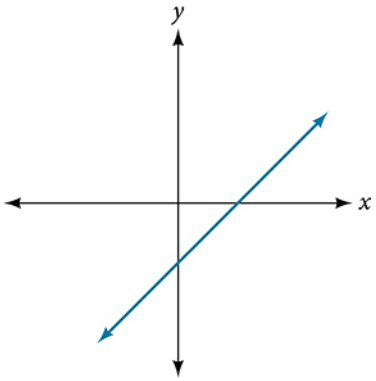
43.



44.



45.



Numeric

For the following exercises, make a table to confirm the end behavior of the function.

46. $f(x) = -x^3$

47. $f(x) = x^4 - 5x^2$

48. $f(x) = x^2(1-x)^2$

49. $f(x) = (x-1)(x-2)(3-x)$

50. $f(x) = \frac{x^5}{10} - x^4$

Technology

For the following exercises, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

51. $f(x) = x^3(x-2)$

52. $f(x) = x(x-3)(x+3)$

53. $f(x) = x(14-2x)(10-2x)$

54. $f(x) = x(14 - 2x)(10 - 2x)^2$ **55.** $f(x) = x^3 - 16x$ **56.** $f(x) = x^3 - 27$

57. $f(x) = x^4 - 81$ **58.** $f(x) = -x^3 + x^2 + 2x$ **59.** $f(x) = x^3 - 2x^2 - 15x$

60. $f(x) = x^3 - 0.01x$

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

61. The y- intercept is $(0, -4)$. The x- intercepts are $(-2, 0)$, $(2, 0)$. Degree is 2.
End behavior: as $x \to -\infty$, $f(x) \to \infty$; as $x \to \infty$, $f(x) \to \infty$.

62. The y- intercept is $(0, 9)$. The x- intercepts are $(-3, 0)$, $(3, 0)$. Degree is 2.
End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$.

63. The y- intercept is $(0, 0)$. The x- intercepts are $(0, 0)$, $(2, 0)$. Degree is 3.
End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

64. The y- intercept is $(0, 1)$. The x- intercept is $(1, 0)$. Degree is 3.
End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.

65. The y- intercept is $(0, 1)$. There is no x- intercept. Degree is 4.
End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$.

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

66. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.

67. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.

68. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.

69. An open box is to be constructed by cutting out square corners of x- inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.

70. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

5.3 Graphs of Polynomial Functions

Learning Objectives

In this section, you will:

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 1.

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

Table 1

The revenue can be modeled by the polynomial function

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$$

where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure 1 shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

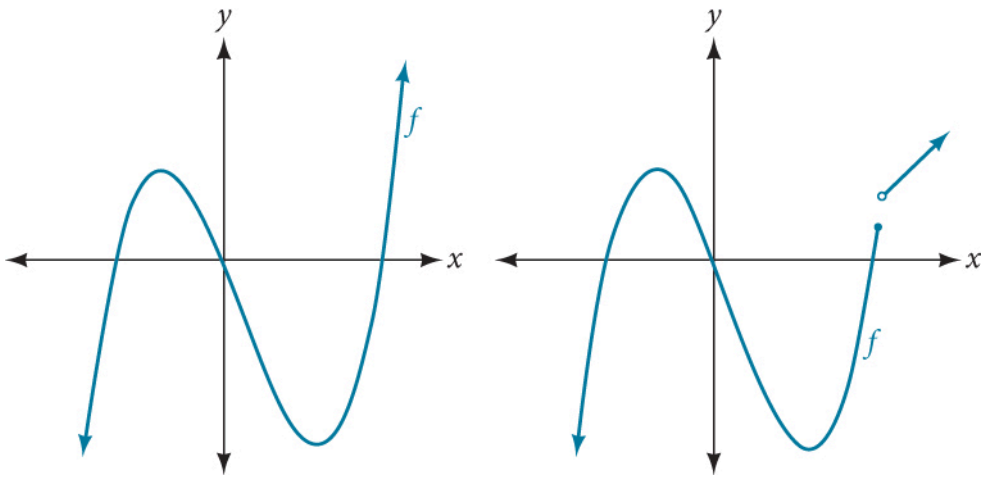


Figure 1

EXAMPLE 1

Recognizing Polynomial Functions

Which of the graphs in Figure 2 represents a polynomial function?

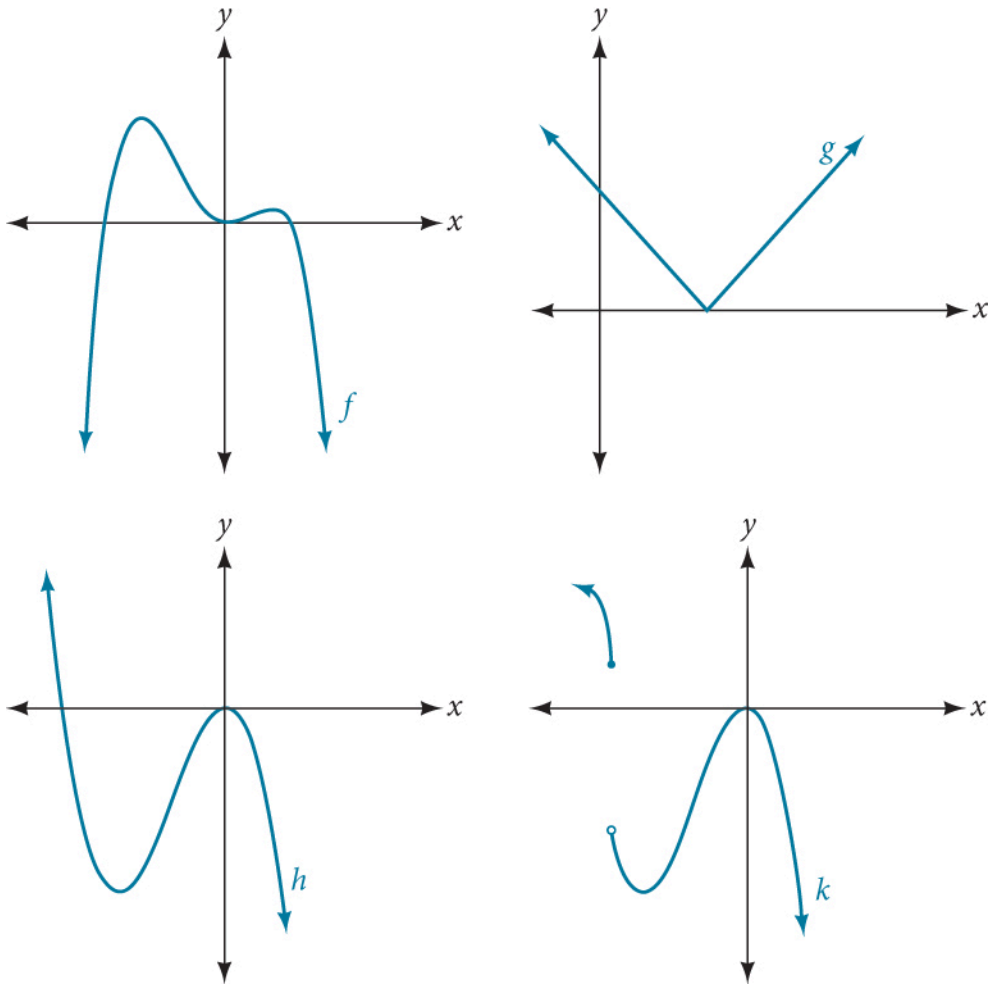


Figure 2

Solution

The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Q&A **Do all polynomial functions have as their domain all real numbers?**

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x- intercepts because at the x- intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

HOW TO

Given a polynomial function f, find the *x*-intercepts by factoring.

1. Set $f(x) = 0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x- intercepts.

EXAMPLE 2

Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the *x*-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$\begin{aligned} x^6 - 3x^4 + 2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2(x^4 - 3x^2 + 2) &= 0 && \text{Factor the trinomial.} \\ x^2(x^2 - 1)(x^2 - 2) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x^2 &= 0 \quad \text{or} \quad (x^2 - 1) = 0 \quad \text{or} \quad (x^2 - 2) = 0 \\ x &= 0 \quad\quad\quad x^2 = 1 \quad\quad\quad x^2 = 2 \\ &\quad\quad\quad\;\; x = \pm 1 \quad\quad\quad x = \pm\sqrt{2} \end{aligned}$$

This gives us five *x*-intercepts: $(0,0), (1,0), (-1,0), (\sqrt{2},0)$, and $(-\sqrt{2},0)$. See Figure 3. We can see that this is an even function because it is symmetric about the *y*-axis.

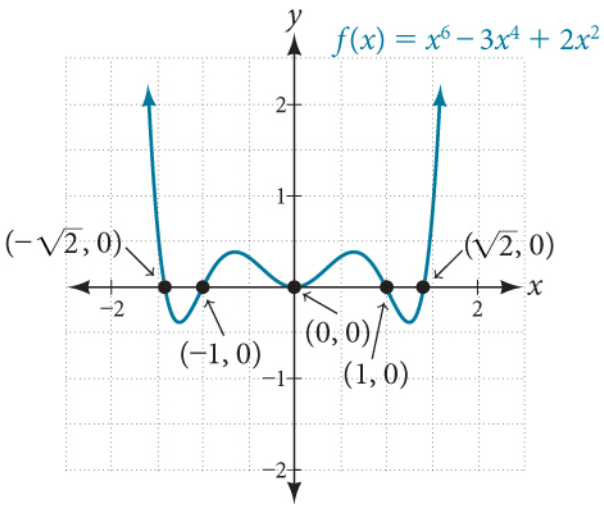


Figure 3

EXAMPLE 3

Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the *x*-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2 - 1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \quad \text{or} \quad x-1 = 0 \quad \text{or} \quad x-5 = 0 \\ x &= -1 \qquad\quad x = 1 \qquad\quad x = 5 \end{aligned}$$

There are three *x*-intercepts: $(-1, 0), (1, 0)$, and $(5, 0)$. See Figure 4.

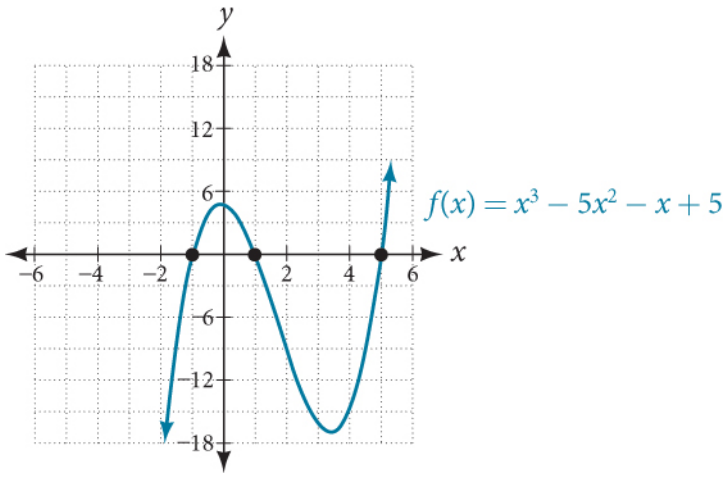


Figure 4

EXAMPLE 4

Finding the *y*- and *x*-Intercepts of a Polynomial in Factored Form

Find the *y*- and *x*-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution

The *y*-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the *y*-intercept is $(0, 12)$.

The *x*-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0$$

$$\begin{aligned} (x-2)^2 &= 0 \qquad\quad (2x+3) = 0 \\ x-2 &= 0 \quad \text{or} \quad x = -\tfrac{3}{2} \\ x &= 2 \end{aligned}$$

So the *x*-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure 5.

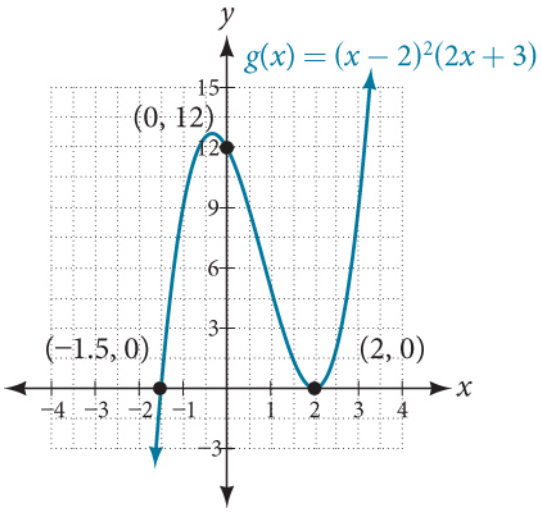


Figure 5

EXAMPLE 5

Finding the *x*-Intercepts of a Polynomial Function Using a Graph

Find the *x*-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 6, it appears that there are *x*-intercepts at $x = -3, -2$, and 1.

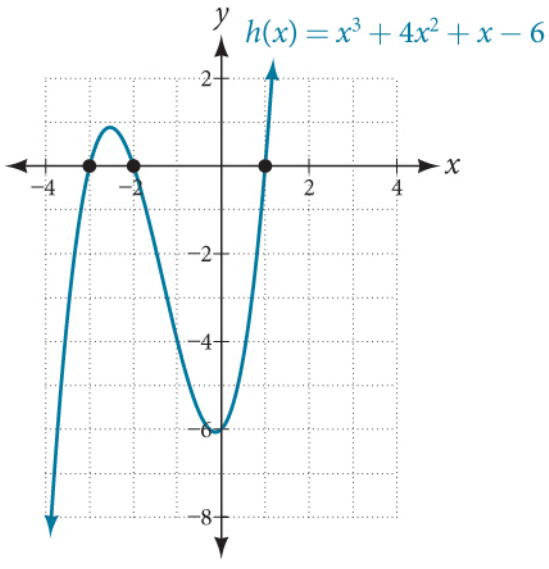


Figure 6

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each *x*-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3 + 4x^2 + x - 6 \\ &= (x + 3)(x + 2)(x - 1) \end{aligned}$$

TRY IT #1 Find the *y*- and *x*-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Identifying Zeros and Their Multiplicities

Graphs behave differently at various *x*-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and "bounce" off.

Suppose, for example, we graph the function shown.

$$f(x) = (x + 3)(x - 2)^2(x + 1)^3$$

Notice in Figure 7 that the behavior of the function at each of the *x*-intercepts is different.

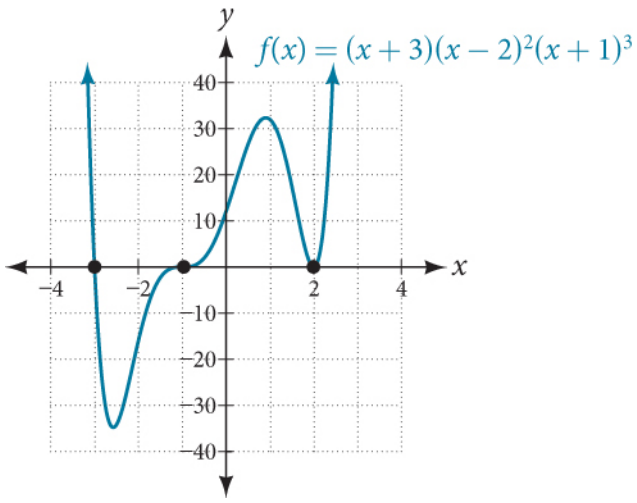


Figure 7 Identifying the behavior of the graph at an *x*-intercept by examining the multiplicity of the zero.

The *x*-intercept $x = -3$ is the solution of equation $(x + 3) = 0$. The graph passes directly through the *x*-intercept at $x = -3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The *x*-intercept $x = 2$ is the repeated solution of equation $(x - 2)^2 = 0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x - 2)^2 = (x - 2)(x - 2)$$

The factor is repeated, that is, the factor $(x - 2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x = 2$, has multiplicity 2 because the factor $(x - 2)$ occurs twice.

The *x*-intercept $x = -1$ is the repeated solution of factor $(x + 1)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

For zeros with even multiplicities, the graphs *touch* or are tangent to the *x*-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the *x*-axis. See Figure 8 for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

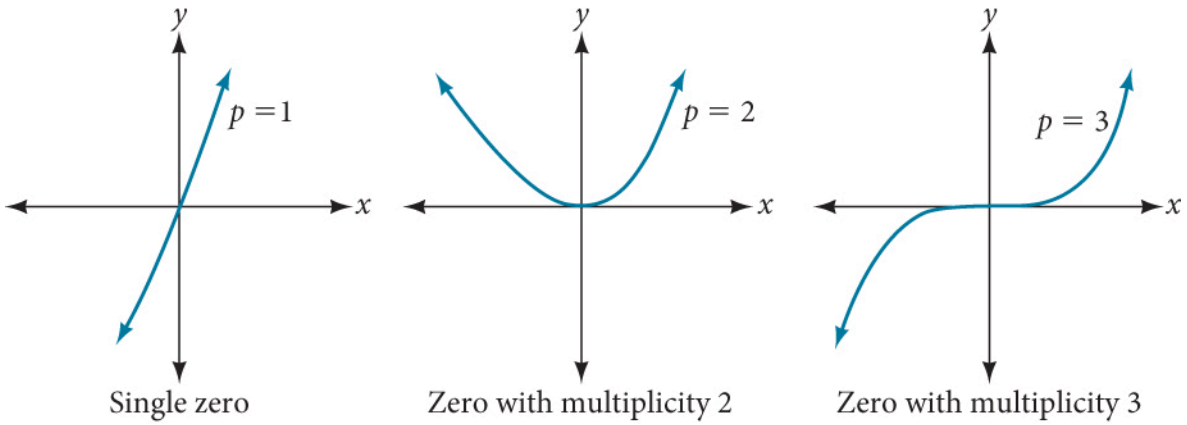


Figure 8

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the *x*-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the *x*-axis.

Graphical Behavior of Polynomials at *x*-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x- intercept h is determined by the power p. We say that $x=h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch the *x*-axis at zeros with even multiplicities. The graph will cross the *x*-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

HOW TO

Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities.

1. If the graph crosses the *x*-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the *x*-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the *x*-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is $\leq n$.

EXAMPLE 6

Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure 9 to identify the zeros of the function and their possible multiplicities.

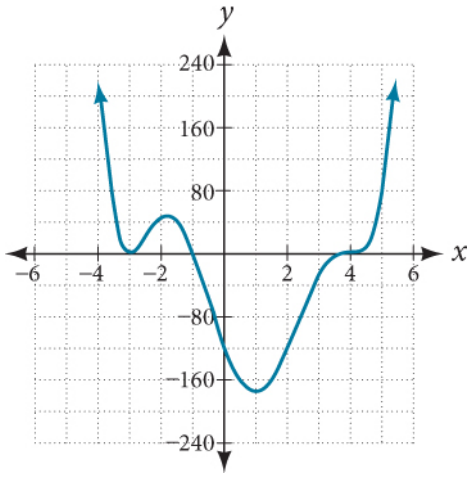


Figure 9

Solution

The polynomial function is of degree 6. The sum of the multiplicities must be 6.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the *x*-axis, so the multiplicity of the zero must be even. The zero of -3 most likely has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the *x*-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is 6.

TRY IT #2 Use the graph of the function of degree 9 in Figure 10 to identify the zeros of the function and their multiplicities.

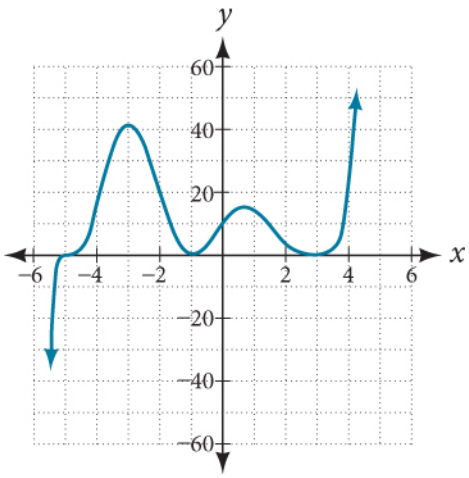


Figure 10

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the *end behavior* of a function. As we pointed out when discussing quadratic equations,

when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure 11 summarizes all four cases.

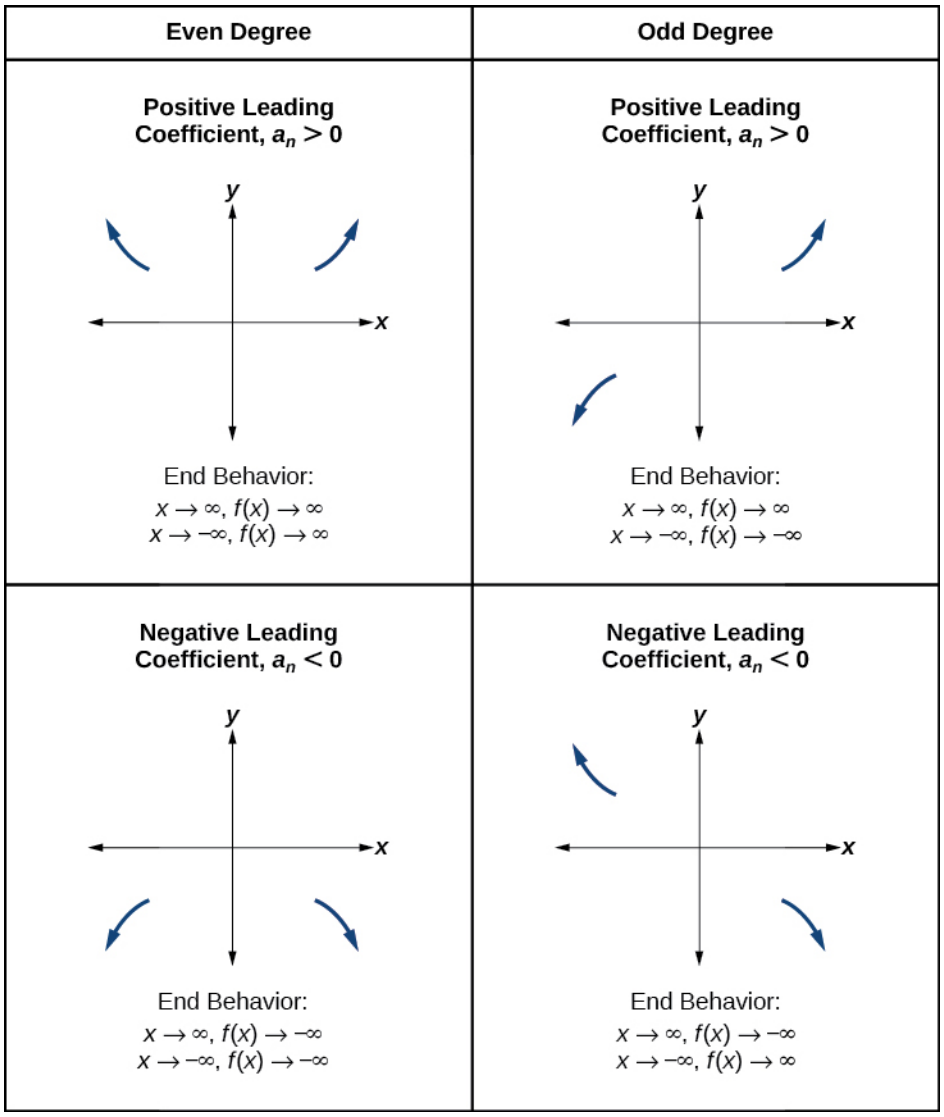


Figure 11

Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure 12. The graph has three turning points.

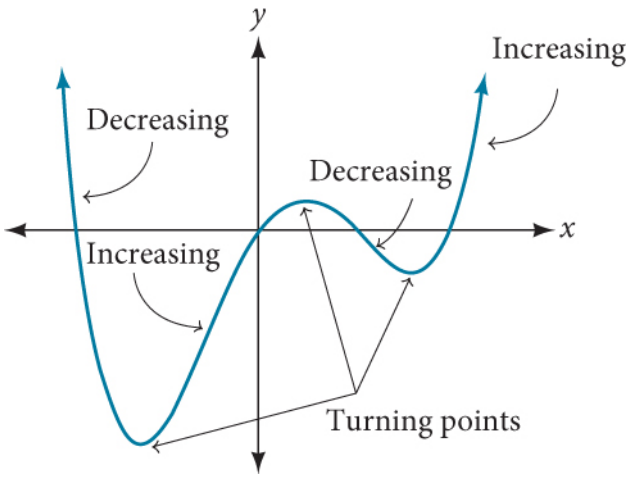


Figure 12

This function f is a 4^{th} degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Interpreting Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).

A polynomial of degree n will have at most $n-1$ turning points.

EXAMPLE 7

Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

ⓐ $f(x)=-x^3+4x^5-3x^2+1$ ⓑ $f(x)=-(x-1)^2\left(1+2x^2\right)$

Solution

ⓐ

First, rewrite the polynomial function in descending order: $f(x)=4x^5-x^3-3x^2+1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5-1=4$.

ⓑ

First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\ (1+2x^2)$$
$$a_n = -(x^2)\ (2x^2) - 2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4-1=3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

HOW TO

Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y- axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x- intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

EXAMPLE 8

Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two *x*-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this *x*-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The *y*-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The *y*-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 13.

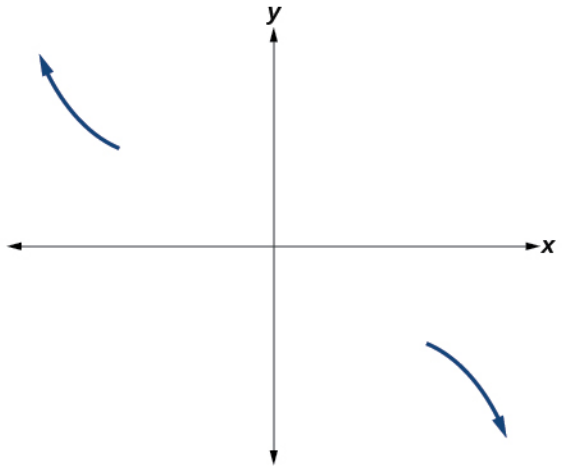


Figure 13

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$, so we know the graph starts in the second quadrant and is decreasing toward the x- axis.

- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of the *x*-axis, so the function must start increasing.
  At $(0, 90)$, the graph crosses the *y*-axis at the *y*-intercept. See Figure 14.

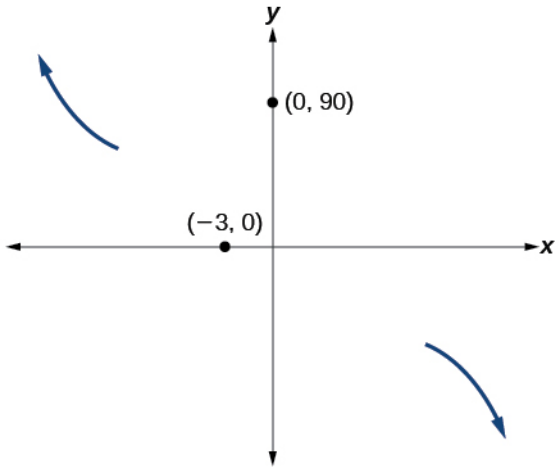


Figure 14

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure 15.

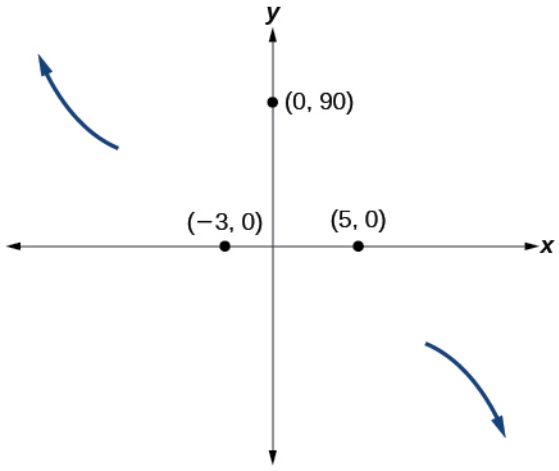


Figure 15

As $x \to \infty$ the function $f(x) \to -\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 16, and verify that the resulting graph looks like our sketch in Figure 15.

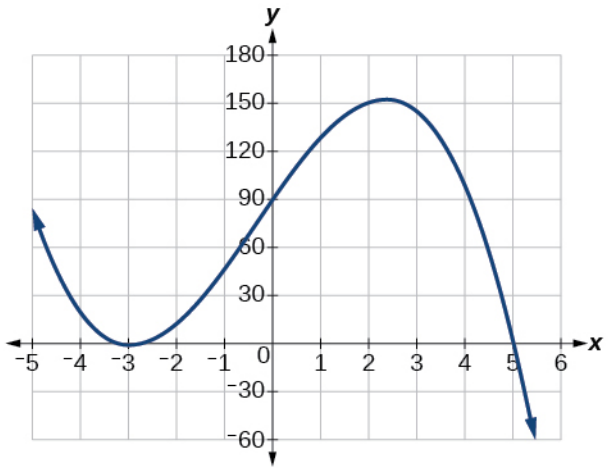


Figure 16 The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$

TRY IT #3 Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the *x*-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$. (While the theorem is intuitive, the proof is actually quite complicated and requires higher mathematics.) We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x- axis and another point at $x = b$ lies below the x- axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x- axis. Call this point $(c, \ f(c))$. This means that we are assured there is a solution c where $f(c) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x- axis. Figure 17 shows that there is a zero between a and b.

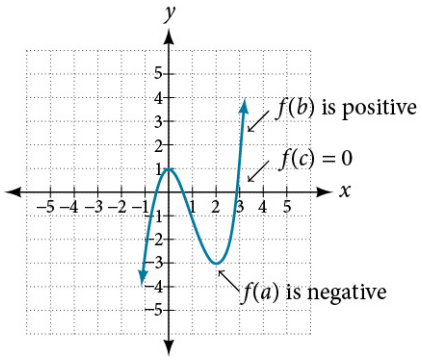


Figure 17 Using the Intermediate Value Theorem to show there exists a zero.

Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

EXAMPLE 9

Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution

As a start, evaluate $f(x)$ at the integer values $x = 1, 2, 3$, and 4. See Table 2.

x	1	2	3	4
$f(x)$	5	0	–3	2

Table 2

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

Analysis

We can also see on the graph of the function in Figure 18 that there are two real zeros between $x = 1$ and $x = 4$.

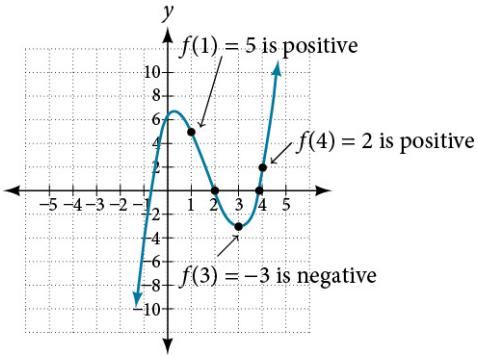


Figure 18

TRY IT #4 Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a polynomial function written in factored form will have an *x*-intercept where each factor is equal to zero, we can form a function that will pass through a set of *x*-intercepts by introducing a corresponding set of factors.

Factored Form of Polynomials

If a polynomial of degree p that can be factored into strictly linear factors has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the *x*-intercept.

HOW TO

Given a graph of a polynomial function, write a formula for the function.

1. Identify the *x*-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the *x*-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the *y*-intercept may be easiest) to determine the stretch factor.

EXAMPLE 10

Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 19.

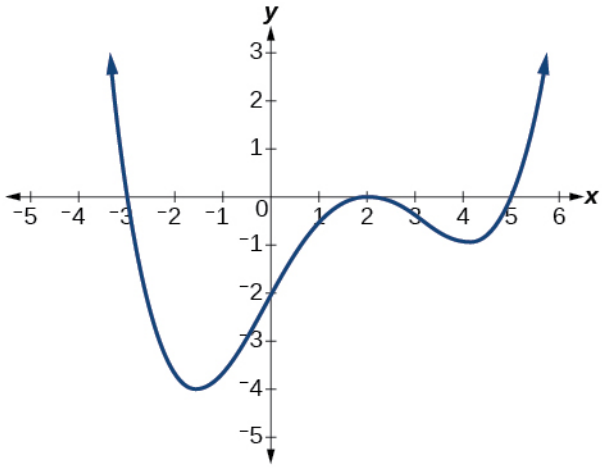


Figure 19

Solution

This graph has three *x*-intercepts: $x = -3, 2$, and 5. The *y*-intercept is located at $(0, -2)$. At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x+3)(x-2)^2(x-5)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y- intercept $(0, -2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \tfrac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

TRY IT #5 Given the graph shown in Figure 20, write a formula for the function shown.

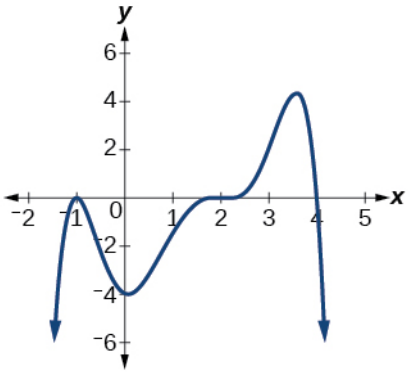


Figure 20

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

Local and Global Extrema

A local maximum or local minimum at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$. If a function has a local maximum at a, then $f(a) \geq f(x)$ for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$ for all x in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in Figure 21.

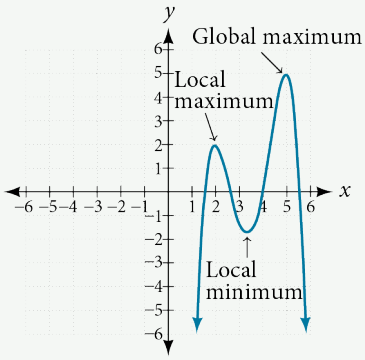


Figure 21

Q&A **Do all polynomial functions have a global minimum or maximum?**

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

EXAMPLE 11

Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic and then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution

We will start this problem by drawing a picture like that in Figure 22, labeling the width of the cut-out squares with a variable, w.

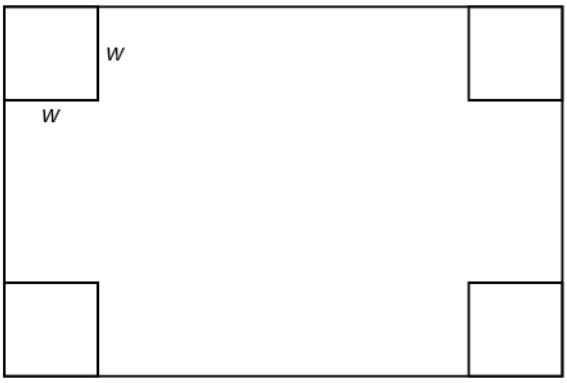


Figure 22

Notice that after a square is cut out from each end, it leaves a $(14-2w)$ cm by $(20-2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

$$\begin{aligned} V(w) &= (20-2w)(14-2w)w \\ &= 280w-68w^2+4w^3 \end{aligned}$$

Notice, since the factors are w, $20-2w$ and $14-2w$, the three zeros are 10, 7, and 0, respectively. Because a height of 0 cm is not reasonable, we consider the only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0<w<7$. Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure 23. We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem—values from 0 to 7.

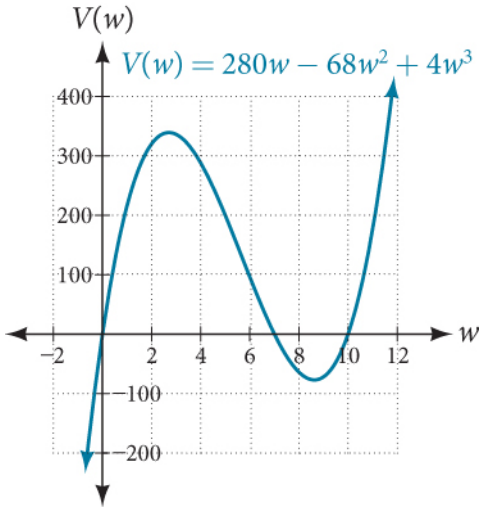


Figure 23

From this graph, we turn our focus to only the portion on the reasonable domain, $[0,\ 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure 24.

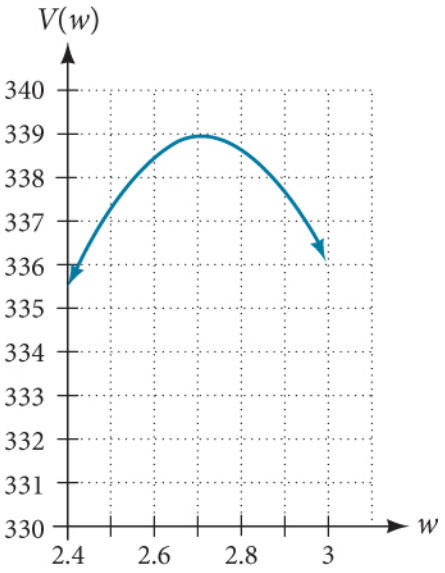


Figure 24

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

TRY IT #6 Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

MEDIA

Access the following online resource for additional instruction and practice with graphing polynomial functions.

Intermediate Value Theorem (https://openstax.org/l/ivt)

5.3 SECTION EXERCISES

Verbal

1. What is the difference between an x- intercept and a zero of a polynomial function f?

2. If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?

3. Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

4. Explain how the factored form of the polynomial helps us in graphing it.

5. If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Algebraic

For the following exercises, find the x- or t-intercepts of the polynomial functions.

6. $C(t) = 2(t-4)(t+1)(t-6)$

7. $C(t) = 3(t+2)(t-3)(t+5)$

8. $C(t) = 4t(t-2)^2(t+1)$

9. $C(t) = 2t(t-3)(t+1)^2$

10. $C(t) = 2t^4 - 8t^3 + 6t^2$

11. $C(t) = 4t^4 + 12t^3 - 40t^2$

12. $f(x) = x^4 - x^2$

13. $f(x) = x^3 + x^2 - 20x$

14. $f(x) = x^3 + 6x^2 - 7x$

15. $f(x) = x^3 + x^2 - 4x - 4$

16. $f(x) = x^3 + 2x^2 - 9x - 18$

17. $f(x) = 2x^3 - x^2 - 8x + 4$

18. $f(x) = x^6 - 7x^3 - 8$

19. $f(x) = 2x^4 + 6x^2 - 8$

20. $f(x) = x^3 - 3x^2 - x + 3$

21. $f(x) = x^6 - 2x^4 - 3x^2$

22. $f(x) = x^6 - 3x^4 - 4x^2$

23. $f(x) = x^5 - 5x^3 + 4x$

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

24. $f(x) = x^3 - 9x$, between $x = -4$ and $x = -2$.

25. $f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.

26. $f(x) = x^5 - 2x$, between $x = 1$ and $x = 2$.

27. $f(x) = -x^4 + 4$, between $x = 1$ and $x = 3$.

28. $f(x) = -2x^3 - x$, between $x = -1$ and $x = 1$.

29. $f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

For the following exercises, find the zeros and give the multiplicity of each.

30. $f(x) = (x+2)^3(x-3)^2$

31. $f(x) = x^2(2x+3)^5(x-4)^2$

32. $f(x) = x^3(x-1)^3(x+2)$

33. $f(x) = x^2\left(x^2 + 4x + 4\right)$

34. $f(x) = (2x+1)^3\left(9x^2 - 6x + 1\right)$

35. $f(x) = (3x+2)^5\left(x^2 - 10x + 25\right)$

36. $f(x) = x\left(4x^2 - 12x + 9\right)\left(x^2 + 8x + 16\right)$

37. $f(x) = x^6 - x^5 - 2x^4$

38. $f(x) = 3x^4 + 6x^3 + 3x^2$

39. $f(x) = 4x^5 - 12x^4 + 9x^3$

40. $f(x) = 2x^4\left(x^3 - 4x^2 + 4x\right)$

41. $f(x) = 4x^4\left(9x^4 - 12x^3 + 4x^2\right)$

Graphical

For the following exercises, graph the polynomial functions. Note x- and y- intercepts, multiplicity, and end behavior.

42. $f(x) = (x+3)^2(x-2)$

43. $g(x) = (x+4)(x-1)^2$

44. $h(x) = (x-1)^3(x+3)^2$

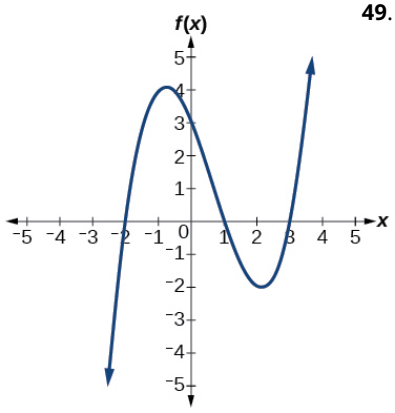
45. $k(x) = (x-3)^3(x-2)^2$

46. $m(x) = -2x(x-1)(x+3)$

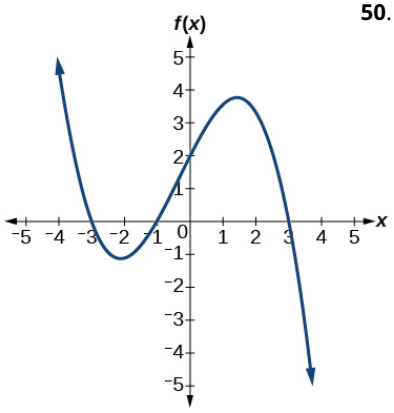
47. $n(x) = -3x(x+2)(x-4)$

For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

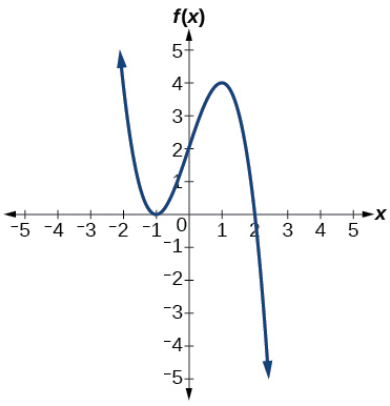
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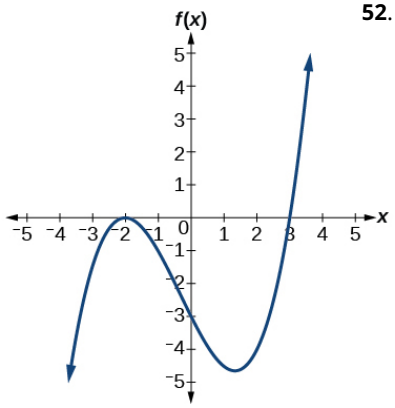
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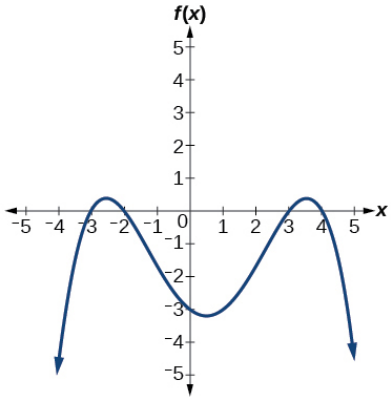
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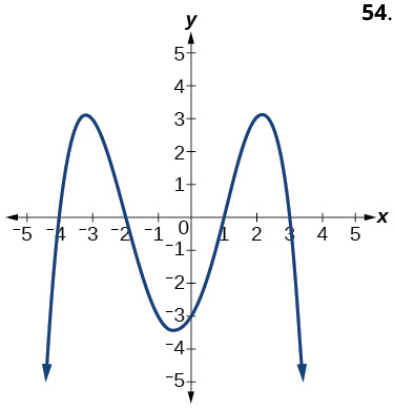


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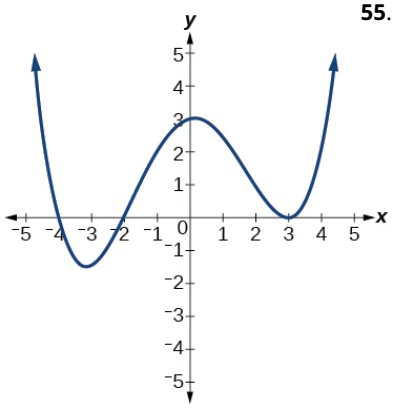


For the following exercises, use the graph to identify zeros and multiplicity.

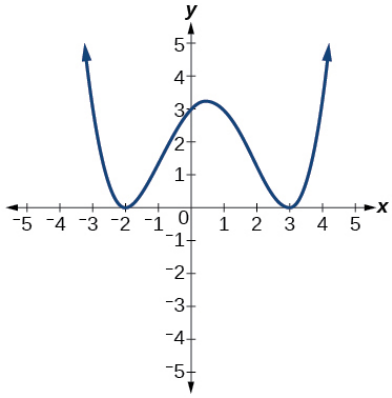
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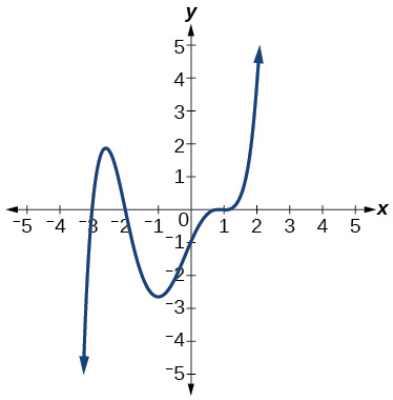
54.



55.



56.



For the following exercises, use the given information about the polynomial graph to write the equation.

57. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. y-intercept at $(0, -4)$.

58. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. y-intercept at $(0, 6)$

59. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. y-intercept at $(0, 9)$

60. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. y-intercept at $(0, -3)$.

61. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$.

62. Degree 3. Zeros at $x = 4$, $x = 3$, and $x = 2$. y-intercept at $(0, -24)$.

63. Degree 3. Zeros at $x = -3$, $x = -2$ and $x = 1$. *y*-intercept at $(0, 12)$.

64. Degree 5. Roots of multiplicity 2 at $x = -3$ and $x = 2$ and a root of multiplicity 1 at $x = -2$. *y*-intercept at $(0,\ 4)$.

65. Degree 4. Roots of multiplicity 2 at $x = \frac{1}{2}$ and roots of multiplicity 1 at $x = 6$ and $x = -2$. *y*-intercept at $(0,18)$.

66. Double zero at $x = -3$ and triple zero at $x = 0$. Passes through the point $(1, 32)$.

Technology

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

67. $f(x) = x^3 - x - 1$

68. $f(x) = 2x^3 - 3x - 1$

69. $f(x) = x^4 + x$

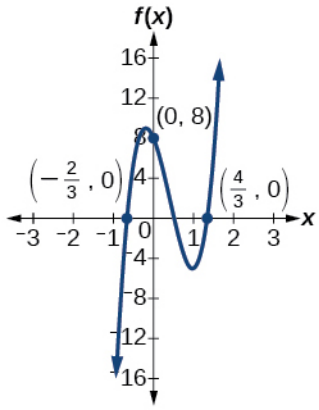
70. $f(x) = -x^4 + 3x - 2$

71. $f(x) = x^4 - x^3 + 1$

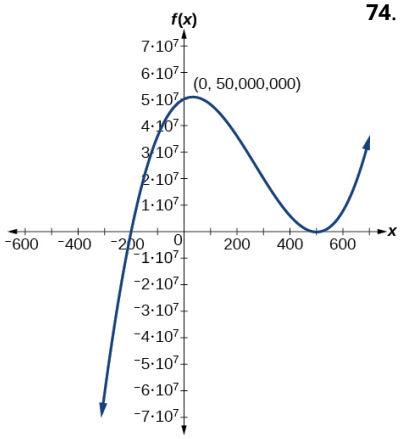
Extensions

For the following exercises, use the graphs to write a polynomial function of least degree.

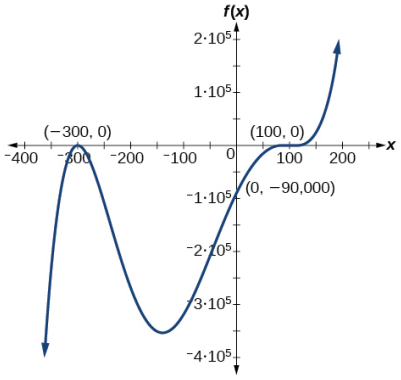
72.



73.



74.



Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

75. A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.

76. Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.

77. A square has sides of 12 units. Squares $x + 1$ by $x + 1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.

78. A cylinder has a radius of $x + 2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.

79. A right circular cone has a radius of $3x + 6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius r and height h.

5.4 Dividing Polynomials

Learning Objectives

In this section, you will:

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.



Figure 1 Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m.[1] We can easily find the volume using elementary geometry.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73{,}800 \end{aligned}$$

So the volume is 73,800 cubic meters (m^3). Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73{,}800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any, or all, of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

1 National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historyculture/lincoln-memorial-building-statistics.htm. Accessed 4/3/2014

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ -15\downarrow \\ \hline 28 \\ -27 \\ \hline 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor} \cdot \text{quotient}) + \text{remainder} \\ 178 &= (3 \cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$\begin{array}{r} \\ x+2\overline{)2x^3 - 3x^2 + 4x + 5} \end{array}$$
Set up the division problem.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3 - 3x^2 + 4x + 5} \end{array}$$
$2x^3$ divided by x is $2x^2$.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3 - 3x^2 + 4x + 5} \\ -(2x^3 + 4x^2) \\ \hline -7x^2 + 4x \end{array}$$
Multiply $x + 2$ by $2x^2$.
Subtract.
Bring down the next term.

$$\begin{array}{r} 2x^2 - 7x \\ x+2\overline{)2x^3 - 3x^2 + 4x + 5} \\ -(2x^3 + 4x^2) \\ \hline -7x^2 + 4x \\ -(-7x^2 - 14x) \\ \hline 18x + 5 \end{array}$$
$-7x^2$ divided by x is $-7x$.
Multiply $x + 2$ by $-7x$.
Subtract. Bring down the next term.

$$\begin{array}{r} 2x^2 - 7x + 18 \\ x+2\overline{)2x^3 - 3x^2 + 4x + 5} \\ -(2x^3 + 4x^2) \\ \hline -7x^2 + 4x \\ -(-7x^2 - 14x) \\ \hline 18x + 5 \\ -(18x + 36) \\ \hline -31 \end{array}$$
$18x$ divided by x is 18.
Multiply $x + 2$ by 18.
Subtract.

We have found

$$\frac{2x^3 - 3x^2 + 4x + 5}{x + 2} = 2x^2 - 7x + 18 - \frac{31}{x + 2}$$

or

$$2x^3 - 3x^2 + 4x + 5 = (x + 2)(2x^2 - 7x + 18) - 31$$

We can identify the dividend, the divisor, the quotient, and the remainder.

$$2x^3 - 3x^2 + 4x + 5 = (x + 2)(2x^2 - 7x + 18) + (-31)$$

Dividend Divisor Quotient Remainder

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The **Division Algorithm** states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

HOW TO

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

EXAMPLE 1

Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution

$x + 1 \overline{)\, 5x^2 + 3x - 2}$ Set up the division problem.

$\quad\;\; 5x$

$x + 1 \overline{)\, 5x^2 + 3x - 2}$ $5x^2$ divided by x is $5x$.

$\quad\;\; 5x$

$x + 1 \overline{)\, 5x^2 + 3x - 2}$

$\quad -(5x^2 + 5x)$ Multiply $x + 1$ by $5x$.

$\quad\quad -2x - 2$ Subtract. Bring down the next term.

$\quad\;\; 5x - 2$

$x + 1 \overline{)\, 5x^2 + 3x - 2}$ $-2x$ divided by x is -2.

$\quad -(5x^2 + 5x)$

$\quad\quad -2x - 2$

$\quad\quad -(-2x - 2)$ Multiply $x + 1$ by -2.

$\quad\quad\quad\;\; 0$ Subtract.

The quotient is $5x - 2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1}=5x-2$$

or

$$5x^2+3x-2=(x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

EXAMPLE 2

Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3+11x^2-31x+15$ by $3x-2$.

Solution

$$\begin{array}{rl}
2x^2+5x-7 & \\
3x-2\overline{)\,6x^3+11x^2-31x+15} & 6x^3 \text{ divided by } 3x \text{ is } 2x^2.\\
\underline{-(6x^3-4x^2)} & \text{Multiply } 3x-2 \text{ by } 2x^2.\\
15x^2-31x & \text{Subtract. Bring down the next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x.\\
\underline{-(15x^2-10x)} & \text{Multiply } 3x-2 \text{ by } 5x.\\
-21x+15 & \text{Subtract. Bring down the next term. } -21x \text{ divided by } 3x \text{ is } -7.\\
\underline{-(-21x+14)} & \text{Multiply } 3x-2 \text{ by } -7.\\
1 & \text{Subtract. The remainder is 1.}
\end{array}$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

TRY IT #1 Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$
\begin{array}{r}
2x^2 - 7x + 18 \\
x + 2 \overline{)\, 2x^3 - 3x^2 + 4x + 5} \\
\underline{-(2x^3 + 4x^2)} \\
-7x^2 + 4x \\
\underline{-(-7x^2 - 14x)} \\
18x + 5 \\
\underline{-(18x + 36)} \\
-31
\end{array}
$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$
\begin{array}{r|rrrr}
2 & 2 & -3 & 4 & 5 \\
 & -2 & -4 & & \\
\hline
 & & -7 & 14 & \\
 & & & 18 & -36 \\
\hline
 & & & & -31
\end{array}
$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$
\begin{array}{r|rrrr}
-2 & 2 & -3 & 4 & 5 \\
 & & -4 & 14 & -36 \\
\hline
 & 2 & -7 & 18 & -31
\end{array}
$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 - 7x + 18$ and the remainder is -31. The process will be made more clear in Example 3.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$ where k is a real number. In **synthetic division**, only the coefficients are used in the division process.

HOW TO

Given two polynomials, use synthetic division to divide.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder. The next number from the right has degree 0, the next number has degree 1, and so on.

EXAMPLE 3

Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

3	5	−3	−36

Bring down the lead coefficient. Multiply the lead coefficient by k.

3	5	−3	−36
		15	
	5		

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

3	5	−3	−36
		15	36
	5	12	0

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36$

EXAMPLE 4

Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution

The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

−2	4	10	−6	−20
		−8	−4	20
	4	2	−10	0

The result is $4x^2 + 2x - 10$. The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

Analysis

The graph of the polynomial function $f(x) = 4x^3 + 10x^2 - 6x - 20$ in Figure 2 shows a zero at $x = k = -2$. This confirms that $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

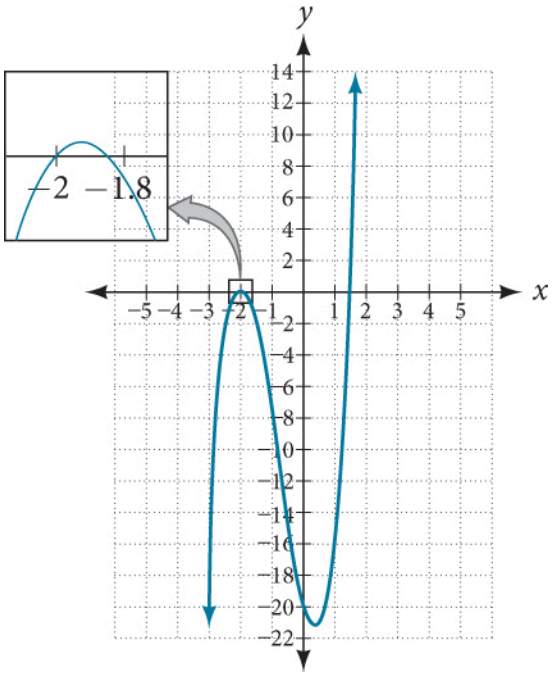


Figure 2

EXAMPLE 5

Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution

Notice there is no *x*-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$.

TRY IT #2 Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

EXAMPLE 6

Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height, h, of the solid.

Solution

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure 3.

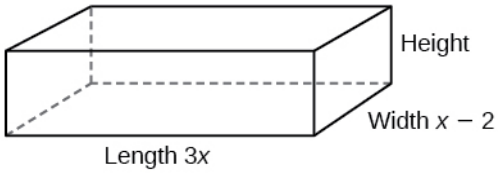


Figure 3

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$\begin{aligned} V &= l \cdot w \cdot h \\ 3x^4 - 3x^3 - 33x^2 + 54x &= 3x \cdot (x-2) \cdot h \end{aligned}$$

To solve for h, first divide both sides by $3x$.

$$\begin{aligned} \frac{3x \cdot (x-2) \cdot h}{3x} &= \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} \\ (x-2)h &= x^3 - x^2 - 11x + 18 \end{aligned}$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

TRY IT #3 The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

MEDIA

Access these online resources for additional instruction and practice with polynomial division.

Dividing a Trinomial by a Binomial Using Long Division (https://openstax.org/l/dividetribild)
Dividing a Polynomial by a Binomial Using Long Division (https://openstax.org/l/dividepolybild)
Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division (https://openstax.org/l/dividepolybisd2)
Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division (https://openstax.org/l/dividepolybisd4)

5.4 SECTION EXERCISES

Verbal

1. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

2. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

3. $\left(x^2 + 5x - 1\right) \div (x - 1)$

4. $\left(2x^2 - 9x - 5\right) \div (x - 5)$

5. $\left(3x^2 + 23x + 14\right) \div (x + 7)$

6. $\left(4x^2 - 10x + 6\right) \div (4x + 2)$

7. $\left(6x^2 - 25x - 25\right) \div (6x + 5)$

8. $\left(-x^2 - 1\right) \div (x + 1)$

9. $\left(2x^2 - 3x + 2\right) \div (x + 2)$

10. $\left(x^3 - 126\right) \div (x - 5)$

11. $\left(3x^2 - 5x + 4\right) \div (3x + 1)$

12. $\left(x^3 - 3x^2 + 5x - 6\right) \div (x - 2)$

13. $\left(2x^3 + 3x^2 - 4x + 15\right) \div (x + 3)$

For the following exercises, use synthetic division to find the quotient. Ensure the equation is in the form required by synthetic division. (Hint: divide the dividend and divisor by the coefficient of the linear term in the divisor.)

14. $(3x^3+2x^2-x+4)\div(x-3)$ **15.** $(2x^3-6x^2-7x+6)\div(x-4)$ **16.** $(6x^3-10x^2-7x-15)\div(x+1)$

17. $(4x^3-12x^2-5x-1)\div(2x+1)$ **18.** $(9x^3-9x^2+18x+5)\div(3x-1)$

19. $(3x^3-2x^2+x-4)\div(x+3)$ **20.** $(-6x^3+x^2-4)\div(2x-3)$ **21.** $(2x^3+7x^2-13x-3)\div(2x-3)$

22. $(3x^3-5x^2+2x+3)\div(x+2)$ **23.** $(4x^3-5x^2+13)\div(x+4)$ **24.** $(x^3-3x+2)\div(x+2)$

25. $(x^3-21x^2+147x-343)\div(x-7)$ **26.** $(x^3-15x^2+75x-125)\div(x-5)$

27. $(9x^3-x+2)\div(3x-1)$ **28.** $(6x^3-x^2+5x+2)\div(3x+1)$

29. $(x^4+x^3-3x^2-2x+1)\div(x+1)$ **30.** $(x^4-3x^2+1)\div(x-1)$

31. $(x^4+2x^3-3x^2+2x+6)\div(x+3)$ **32.** $(x^4-10x^3+37x^2-60x+36)\div(x-2)$

33. $(x^4-8x^3+24x^2-32x+16)\div(x-2)$ **34.** $(x^4+5x^3-3x^2-13x+10)\div(x+5)$

35. $(x^4-12x^3+54x^2-108x+81)\div(x-3)$ **36.** $(4x^4-2x^3-4x+2)\div(2x-1)$

37. $(4x^4+2x^3-4x^2+2x+2)\div(2x+1)$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

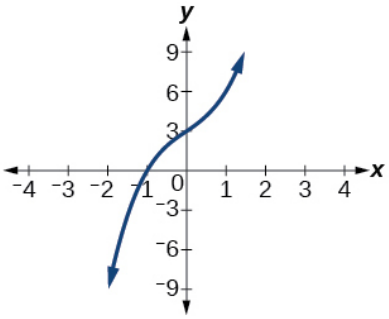
38. $x-2,\ 4x^3-3x^2-8x+4$ **39.** $x-2,\ 3x^4-6x^3-5x+10$ **40.** $x+3,\ -4x^3+5x^2+8$

41. $x-2,\ 4x^4-15x^2-4$ **42.** $x-\frac{1}{2},\ 2x^4-x^3+2x-1$ **43.** $x+\frac{1}{3},\ 3x^4+x^3-3x+1$

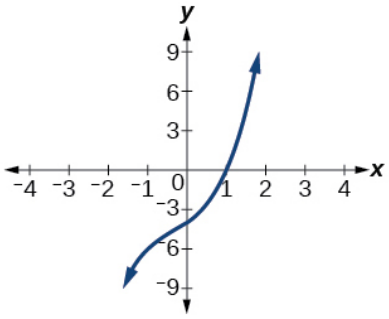
Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

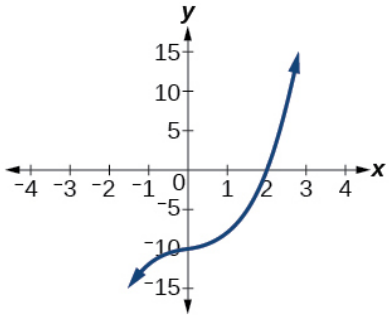
44. Factor is x^2-x+3



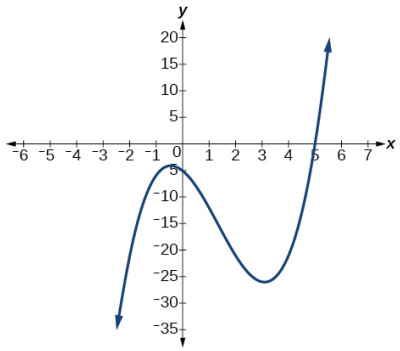
45. Factor is (x^2+2x+4)



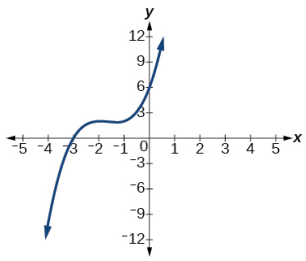
46. Factor is x^2+2x+5



47. Factor is $x^2 + x + 1$



48. Factor is $x^2 + 2x + 2$



For the following exercises, use synthetic division to find the quotient and remainder.

49. $\frac{4x^3-33}{x-2}$

50. $\frac{2x^3+25}{x+3}$

51. $\frac{3x^3+2x-5}{x-1}$

52. $\frac{-4x^3-x^2-12}{x+4}$

53. $\frac{x^4-22}{x+2}$

Technology

For the following exercises, use a calculator with CAS to answer the questions.

54. Consider $\frac{x^k-1}{x-1}$ with $k = 1,\ 2,\ 3$. What do you expect the result to be if $k = 4$?

55. Consider $\frac{x^k+1}{x+1}$ for $k = 1,\ 3,\ 5$. What do you expect the result to be if $k = 7$?

56. Consider $\frac{x^4-k^4}{x-k}$ for $k = 1,\ 2,\ 3$. What do you expect the result to be if $k = 4$?

57. Consider $\frac{x^k}{x+1}$ with $k = 1,\ 2,\ 3$. What do you expect the result to be if $k = 4$?

58. Consider $\frac{x^k}{x-1}$ with $k = 1,\ 2,\ 3$. What do you expect the result to be if $k = 4$?

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

59. $\frac{x+1}{x-i}$

60. $\frac{x^2+1}{x-i}$

61. $\frac{x+1}{x+i}$

62. $\frac{x^2+1}{x+i}$

63. $\frac{x^3+1}{x-i}$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

64. Length is $x + 5$, area is $2x^2 + 9x - 5$.

65. Length is $2x + 5$, area is $4x^3 + 10x^2 + 6x + 15$

66. Length is $3x-4$, area is $6x^4 - 8x^3 + 9x^2 - 9x - 4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

67. Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.

68. Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x-4$, width is $3x-4$.

69. Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x-4$, width is $2x + 3$.

70. Volume is $10x^3 + 30x^2 - 8x - 24$, length is 2, width is $x + 3$.

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

71. Volume is $\pi\left(25x^3 - 65x^2 - 29x - 3\right)$, radius is $5x + 1$.

72. Volume is $\pi\left(4x^3 + 12x^2 - 15x - 50\right)$, radius is $2x + 5$.

73. Volume is $\pi\left(3x^4 + 24x^3 + 46x^2 - 16x - 32\right)$, radius is $x + 4$.

5.5 Zeros of Polynomial Functions

Learning Objectives

In this section, you will:

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.
- Solve real-world applications of polynomial equations

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as well as sheet cakes to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x - k$, the remainder may be found quickly by evaluating the polynomial function at k, that is, $f(k)$ Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

and either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. In practice divisors, $d(x)$ will have degrees less than or equal to the degree of $f(x)$. If the divisor, $d(x)$, is $x - k$, this takes the form

$$f(x) = (x - k)q(x) + r$$

Since the divisor $x - k$ is linear, the remainder will be a constant, r. And, if we evaluate this for $x = k$, we have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= 0 \cdot q(k) + r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$ by $x - k$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is the value $f(k)$.



HOW TO

Given a polynomial function f, evaluate $f(x)$ at $x = k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x - k$.
2. The remainder is the value $f(k)$.

EXAMPLE 1

Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x - 2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array}$$

The remainder is 25. Therefore, $f(2) = 25$.

Analysis

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \\ &= 25 \end{aligned}$$

TRY IT #1 Use the Remainder Theorem to evaluate $f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2$ at $x = -3$.

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x) = (x - k)q(x) + r$$

If k is a zero, then the remainder r is $f(k) = 0$ and $f(x) = (x - k)q(x) + 0$ or $f(x) = (x - k)q(x)$.

Notice, written in this form, $x - k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x - k$ is a factor of $f(x)$.

Similarly, if $x - k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x) = (x - k)q(x) + r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

The Factor Theorem

According to the **Factor Theorem**, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.

HOW TO

Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial.

1. Use synthetic division to divide the polynomial by $(x - k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x - k)$ and the quadratic quotient.
4. If possible, factor the quadratic.

5. Write the polynomial as the product of factors.

EXAMPLE 2

Using the Factor Theorem to Find the Zeros of a Polynomial Expression

Show that $(x+2)$ is a factor of x^3-6x^2-x+30. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

Solution

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)\left(x^2-8x+15\right)$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5)$$

By the Factor Theorem, the zeros of x^3-6x^2-x+30 are –2, 3, and 5.

TRY IT #2 Use the Factor Theorem to find the zeros of $f(x)=x^3+4x^2-4x-16$ given that $(x-2)$ is a factor of the polynomial.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial

Consider a quadratic function with two zeros, $x=\frac{2}{5}$ and $x=\frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

$x-\frac{2}{5}=0$ or $x-\frac{3}{4}=0$	Set each factor equal to 0.
$5x-2=0$ or $4x-3=0$	Multiply both sides of the equation to eliminate fractions.
$f(x)=(5x-2)(4x-3)$	Create the quadratic function, multiplying the factors.
$f(x)=20x^2-23x+6$	Expand the polynomial.
$f(x)=(5\cdot 4)x^2-23x+(2\cdot 3)$	

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

The Rational Zero Theorem

The **Rational Zero Theorem** states that, if the polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ has integer

coefficients $a_n \neq 0$, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.



HOW TO

Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

EXAMPLE 3

Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Solution

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\ \frac{1}{1},\ \pm\ \frac{1}{2} \qquad \frac{p}{q} = \pm\ \frac{2}{1}, \pm\ \frac{2}{2} \qquad \frac{p}{q} = \pm\ \frac{4}{1}, \pm\ \frac{4}{2}$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm 1, \pm 2, \pm 4, \pm \frac{1}{2}$$

EXAMPLE 4

Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of 1}}{\text{factor of 2}} \end{aligned}$$

The factors of 1 are ± 1 and the factors of 2 are ± 1 and ± 2. The possible values for $\frac{p}{q}$ are ± 1 and $\pm \frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

$$\begin{aligned} f(-1) &= 2(-1)^3+(-1)^2-4(-1)+1=4 \\ f(1) &= 2(1)^3+(1)^2-4(1)+1=0 \\ f\left(-\tfrac{1}{2}\right) &= 2\left(-\tfrac{1}{2}\right)^3+\left(-\tfrac{1}{2}\right)^2-4\left(-\tfrac{1}{2}\right)+1=3 \\ f\left(\tfrac{1}{2}\right) &= 2\left(\tfrac{1}{2}\right)^3+\left(\tfrac{1}{2}\right)^2-4\left(\tfrac{1}{2}\right)+1=-\tfrac{1}{2} \end{aligned}$$

Of those, $-1, -\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

TRY IT #3 Use the Rational Zero Theorem to find the rational zeros of $f(x)=x^3-5x^2+2x+1$.

Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use synthetic division repeatedly to determine all of the **zeros** of a polynomial function.

HOW TO

Given a polynomial function f, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

EXAMPLE 5

Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x)=4x^3-3x-1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –1 and q is a factor of 4.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } -1}{\text{factor of } 4} \end{aligned}$$

The factors of –1 are ± 1 and the factors of 4 are $\pm 1, \pm 2$, and ± 4. The possible values for $\frac{p}{q}$ are ± 1, $\pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

$$\begin{array}{r|rrrr} 1 & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(4x^2+4x+1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$\begin{aligned} 2x+1 &= 0 \\ x &= -\tfrac{1}{2} \end{aligned}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Analysis

Look at the graph of the function f in Figure 1. Notice, at $x=-0.5$, the graph bounces off the *x*-axis, indicating the even multiplicity (2,4,6...) for the zero -0.5. At $x=1$, the graph crosses the *x*-axis, indicating the odd multiplicity (1,3,5...) for the zero $x=1$.

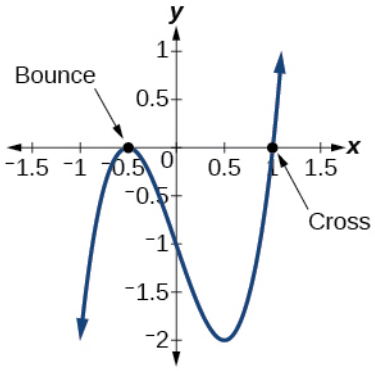


Figure 1

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x)=0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x-c_1$ and a polynomial quotient. Since $x-c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x-c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

The Fundamental Theorem of Algebra

The **Fundamental Theorem of Algebra** states that, if $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n>0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x)=a(x-c_1)(x-c_2)...(x-c_n)$$

where $c_1, c_2, ..., c_n$ are complex numbers. Therefore, $f(x)$ has n roots if we allow for multiplicities.

Q&A **Does every polynomial have at least one imaginary zero?**

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

EXAMPLE 6

Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of 3}}{\text{factor of 3}} \end{aligned}$$

The factors of 3 are ± 1 and ± 3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are $\pm 3, \pm 1$, and $\pm \frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

$$\begin{array}{r|rrrr} -3 & 3 & 9 & 1 & 3 \\ & & -9 & 0 & -3 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

Dividing by $(x + 3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x + 3)\left(3x^2 + 1\right)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$\begin{aligned} 3x^2 + 1 &= 0 \\ x^2 &= -\tfrac{1}{3} \\ x &= \pm\sqrt{-\tfrac{1}{3}} = \pm\tfrac{i\sqrt{3}}{3} \end{aligned}$$

The zeros of $f(x)$ are –3 and $\pm\frac{i\sqrt{3}}{3}$.

Analysis

Look at the graph of the function f in Figure 2. Notice that, at $x = -3$, the graph crosses the *x*-axis, indicating an odd multiplicity (1) for the zero $x = -3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the *x*-intercepts for the function are shown. So either the multiplicity of $x = -3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x = -3$ is three. Either way, our result is correct.

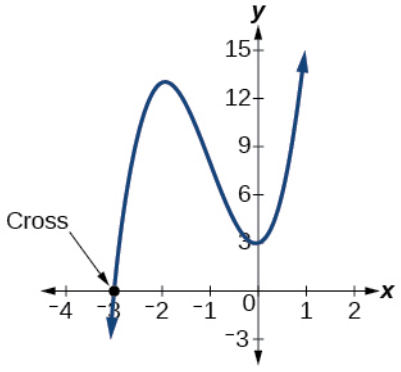


Figure 2

TRY IT #4 Find the zeros of $f(x) = 2x^3 + 5x^2 - 11x + 4$.

Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the Fundamental Theorem of Algebra, as we stated above, is that a polynomial function of degree n

will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a+bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For f to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the Complex Conjugate Theorem.

Complex Conjugate Theorem

According to the **Linear Factorization Theorem,** a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number.

If the polynomial function f has real coefficients and a complex zero in the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero.

HOW TO

Given the zeros of a polynomial function f and a point (*c*, *f*(*c*)) on the graph of f, use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

EXAMPLE 7

Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of –3, 2, i, such that $f(-2)=100$.

Solution

Because $x=i$ is a zero, by the Complex Conjugate Theorem $x=-i$ is also a zero. The polynomial must have factors of $(x+3)$, $(x-2)$, $(x-i)$, and $(x+i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$\begin{aligned} f(x) &= a(x+3)(x-2)(x-i)(x+i) \\ f(x) &= a\left(x^2+x-6\right)\left(x^2+1\right) \\ f(x) &= a\left(x^4+x^3-5x^2+x-6\right) \end{aligned}$$

We need to find *a* to ensure $f(-2)=100$. Substitute $x=-2$ and $f(2)=100$ into $f(x)$.

$$\begin{aligned} 100 &= a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \\ 100 &= a(-20) \\ -5 &= a \end{aligned}$$

So the polynomial function is

$$f(x)=-5\left(x^4+x^3-5x^2+x-6\right)$$

or

$$f(x)=-5x^4-5x^3+25x^2-5x+30$$

Analysis

We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

Q&A **If $2+3i$ were given as a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?**

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

TRY IT #5 Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1)=10$.

Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

Descartes' Rule of Signs

According to **Descartes' Rule of Signs**, if we let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

EXAMPLE 8

Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

Solution

Begin by determining the number of sign changes.

$$f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$$

Figure 3

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the

number of negative real roots.

$$\begin{aligned} f(-x) &= -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12 \\ f(-x) &= -x^4 + 3x^3 + 6x^2 + 4x - 12 \end{aligned}$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

Figure 4

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 1.

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Table 1

Analysis

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 5. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

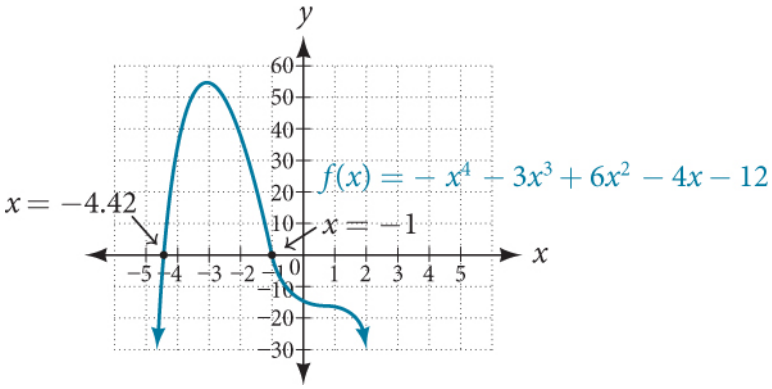


Figure 5

TRY IT #6 Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

EXAMPLE 9

Solving Polynomial Equations

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as

well as sheet cakes to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

Solution

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh$. We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w + 4$. We were given that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w$. Let's write the volume of the cake in terms of width of the cake.

$$\begin{aligned} V &= (w+4)(w)\left(\tfrac{1}{3}w\right) \\ V &= \tfrac{1}{3}w^3 + \tfrac{4}{3}w^2 \end{aligned}$$

Substitute the given volume into this equation.

$$\begin{aligned} 351 &= \tfrac{1}{3}w^3 + \tfrac{4}{3}w^2 && \text{Substitute 351 for } V. \\ 1053 &= w^3 + 4w^2 && \text{Multiply both sides by 3.} \\ 0 &= w^3 + 4w^2 - 1053 && \text{Subtract 1053 from both sides.} \end{aligned}$$

Descartes' rule of signs tells us there is one positive solution. The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1,\ \pm 3,\ \pm 9,\ \pm 13,\ \pm 27,\ \pm 39,\ \pm 81,\ \pm 117,\ \pm 351$, and ± 1053. We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check $x = 1$.

1	1	4	0	−1053
		1	5	5
	1	5	5	−1048

Since 1 is not a solution, we will check $x = 3$.

3	1	4	0	−1053
		3	21	63
	1	7	21	−990

Since 3 is not a solution either, we will test $x = 9$.

9	1	4	0	−1053
		9	117	1053
	1	13	117	0

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

$$l = w + 4 = 9 + 4 = 13 \text{ and } h = \frac{1}{3}w = \frac{1}{3}(9) = 3$$

The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches.

TRY IT #7 A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

MEDIA

Access these online resources for additional instruction and practice with zeros of polynomial functions.

Real Zeros, Factors, and Graphs of Polynomial Functions (https://openstax.org/l/realzeros)
Complex Factorization Theorem (https://openstax.org/l/factortheorem)

Find the Zeros of a Polynomial Function (https://openstax.org/l/findthezeros)
Find the Zeros of a Polynomial Function 2 (https://openstax.org/l/findthezeros2)
Find the Zeros of a Polynomial Function 3 (https://openstax.org/l/findthezeros3)

5.5 SECTION EXERCISES

Verbal

1. Describe a use for the Remainder Theorem.

2. Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

3. What is the difference between rational and real zeros?

4. If Descartes' Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

5. If synthetic division reveals a zero, why should we try that value again as a possible solution?

Algebraic

For the following exercises, use the Remainder Theorem to find the remainder.

6. $\left(x^4-9x^2+14\right)\div(x-2)$

7. $\left(3x^3-2x^2+x-4\right)\div(x+3)$

8. $\left(x^4+5x^3-4x-17\right)\div(x+1)$

9. $\left(-3x^2+6x+24\right)\div(x-4)$

10. $\left(5x^5-4x^4+3x^3-2x^2+x-1\right)\div(x+6)$

11. $\left(x^4-1\right)\div(x-4)$

12. $\left(3x^3+4x^2-8x+2\right)\div(x-3)$

13. $\left(4x^3+5x^2-2x+7\right)\div(x+2)$

For the following exercises, use the given factor and the Factor Theorem to find all real zeros for the given polynomial function.

14. $f(x)=2x^3-9x^2+13x-6;\ \ x-1$

15. $f(x)=2x^3+x^2-5x+2;\ \ x+2$

16. $f(x)=3x^3+x^2-20x+12;\ \ x+3$

17. $f(x)=2x^3+3x^2+x+6;\ \ x+2$

18. $f(x)=-5x^3+16x^2-9;\ \ x-3$

19. $x^3+3x^2+4x+12;\ \ x+3$

20. $4x^3-7x+3;\ \ x-1$

21. $2x^3+5x^2-12x-30,\ \ \ 2x+5$

For the following exercises, use the Rational Zero Theorem to find the real solution(s) to each equation.

22. $x^3-3x^2-10x+24=0$

23. $2x^3+7x^2-10x-24=0$

24. $x^3+2x^2-9x-18=0$

25. $x^3+5x^2-16x-80=0$

26. $x^3-3x^2-25x+75=0$

27. $2x^3-3x^2-32x-15=0$

28. $2x^3+x^2-7x-6=0$

29. $2x^3-3x^2-x+1=0$

30. $3x^3-x^2-11x-6=0$

31. $2x^3-5x^2+9x-9=0$

32. $2x^3-3x^2+4x+3=0$

33. $x^4-2x^3-7x^2+8x+12=0$

34. $x^4+2x^3-9x^2-2x+8=0$

35. $4x^4+4x^3-25x^2-x+6=0$

36. $2x^4-3x^3-15x^2+32x-12=0$

37. $x^4+2x^3-4x^2-10x-5=0$

38. $4x^3-3x+1=0$

39. $8x^4+26x^3+39x^2+26x+6$

For the following exercises, find all complex solutions (real and non-real).

40. $x^3+x^2+x+1=0$

41. $x^3-8x^2+25x-26=0$

42. $x^3+13x^2+57x+85=0$

43. $3x^3-4x^2+11x+10=0$

44. $x^4+2x^3+22x^2+50x-75=0$

45. $2x^3-3x^2+32x+17=0$

Graphical

For the following exercises, use Descartes' Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.

46. $f(x) = x^3 - 1$

47. $f(x) = x^4 - x^2 - 1$

48. $f(x) = x^3 - 2x^2 - 5x + 6$

49. $f(x) = x^3 - 2x^2 + x - 1$

50. $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

51. $f(x) = 2x^3 + 37x^2 + 200x + 300$

52. $f(x) = x^3 - 2x^2 - 16x + 32$

53. $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

54. $f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8$

55. $f(x) = 10x^4 - 21x^2 + 11$

Numeric

For the following exercises, list all possible rational zeros for the functions.

56. $f(x) = x^4 + 3x^3 - 4x + 4$

57. $f(x) = 2x^3 + 3x^2 - 8x + 5$

58. $f(x) = 3x^3 + 5x^2 - 5x + 4$

59. $f(x) = 6x^4 - 10x^2 + 13x + 1$

60. $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$

Technology

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

61. $f(x) = 6x^3 - 7x^2 + 1$

62. $f(x) = 4x^3 - 4x^2 - 13x - 5$

63. $f(x) = 8x^3 - 6x^2 - 23x + 6$

64. $f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$

65. $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

Extensions

For the following exercises, construct a polynomial function of least degree possible using the given information.

66. Real roots: –1, 1, 3 and $(2, f(2)) = (2, 4)$

67. Real roots: –1, 1 (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$

68. Real roots: –2, $\frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$

69. Real roots: $-\frac{1}{2}$, 0, $\frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$

70. Real roots: –4, –1, 1, 4 and $(-2, f(-2)) = (-2, 10)$

Real-World Applications

For the following exercises, find the dimensions of the box described.

71. The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

72. The length, width, and height are consecutive whole numbers. The volume is 120 cubic inches.

73. The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

74. The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.

75. The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

For the following exercises, find the dimensions of the right circular cylinder described.

76. The radius is 3 inches more than the height. The volume is 16π cubic meters.

77. The height is one less than one half the radius. The volume is 72π cubic meters.

78. The radius and height differ by one meter. The radius is larger and the volume is 48π cubic meters.

79. The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.

80. The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

5.6 Rational Functions

Learning Objectives

In this section, you will:

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C(x) = 15{,}000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x.

The average cost function, which yields the average cost per item for x items produced, is

$$f(x) = \frac{15{,}000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 1, and notice some of their features.

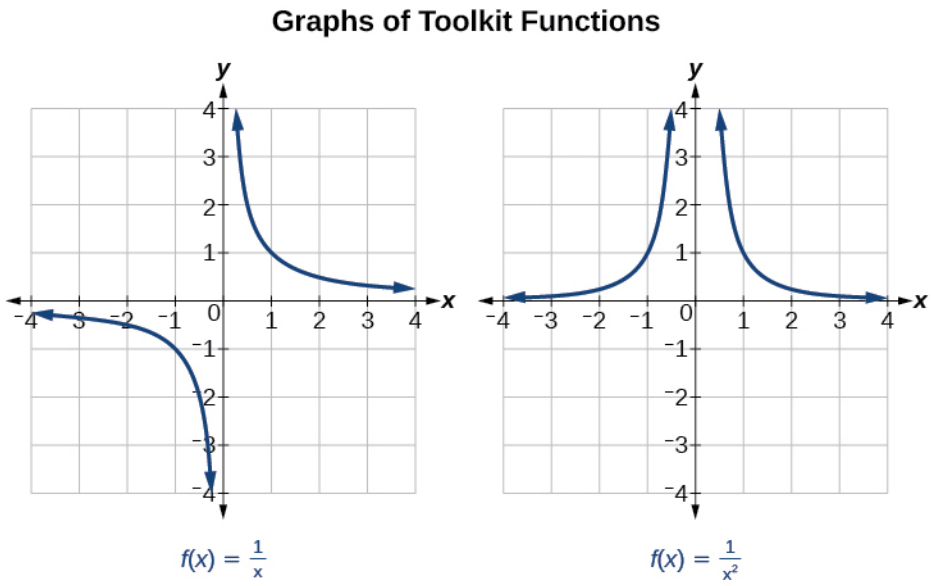


Figure 1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

1. On the left branch of the graph, the curve approaches the *x*-axis $(y = 0)$ as $x \to -\infty$.
2. As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
3. Finally, on the right branch of the graph, the curves approaches the *x*-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value. See Table 1.

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Table 1

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 2.

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

Table 2

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 3.

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

Table 3

We write in arrow notation

$$\text{As } x \to 0^+, \ f(x) \to \infty.$$

See Figure 2.

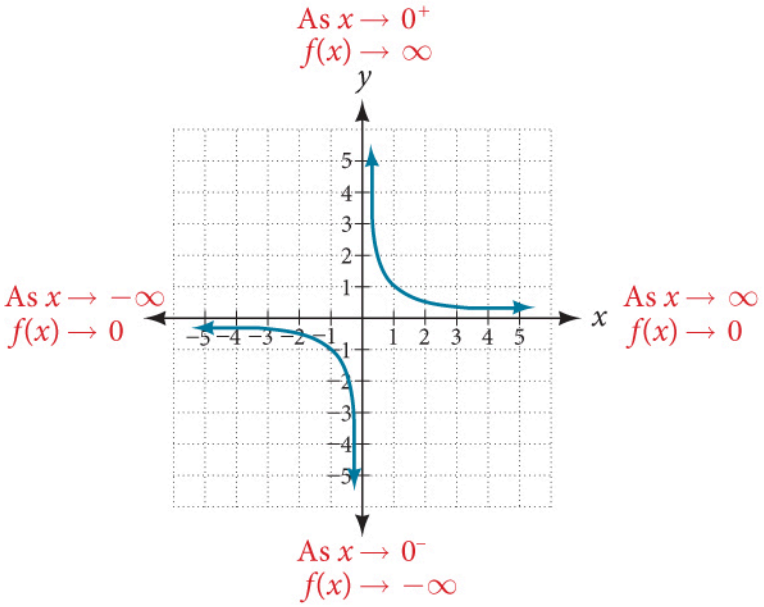


Figure 2

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See Figure 3.

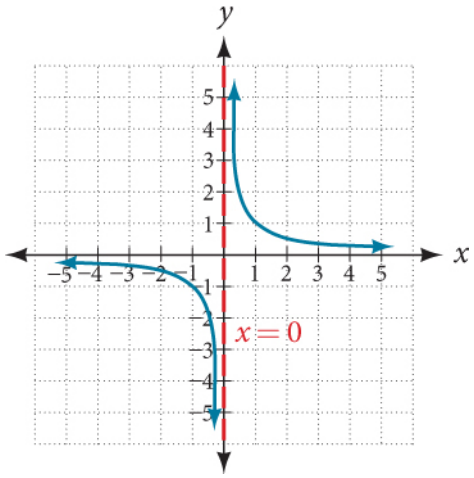


Figure 3

Vertical Asymptote

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the input approaches a from either the left or the right. We write

$$\text{As } x \to a^-, f(x) \to \pm\infty \text{ or } x \to a^+, f(x) \to \pm\infty.$$

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0. See Figure 4. Symbolically, using arrow notation

$$\text{As } x \to \infty, f(x) \to 0, \text{and as } x \to -\infty, f(x) \to 0.$$

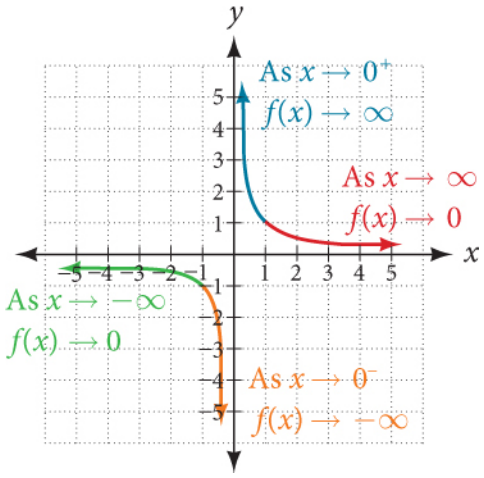


Figure 4

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 5.

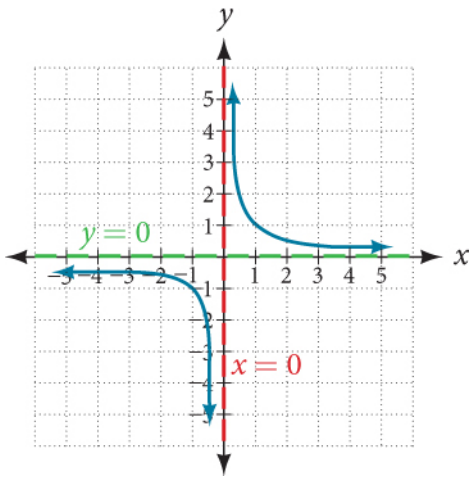


Figure 5

Horizontal Asymptote

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty, \quad f(x) \to b.$$

EXAMPLE 1

Using Arrow Notation

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 6.

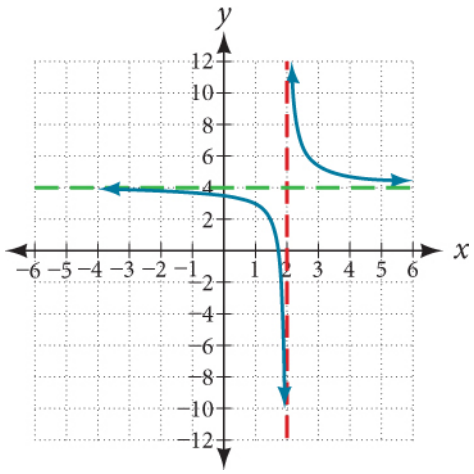


Figure 6

Solution

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-, f(x) \to -\infty, \quad \text{and as } x \to 2^+, \quad f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \to \infty, \quad f(x) \to 4 \text{ and as } x \to -\infty, \quad f(x) \to 4.$$

TRY IT #1 Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

EXAMPLE 2

Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2}$$

The graph of the shifted function is displayed in Figure 7.

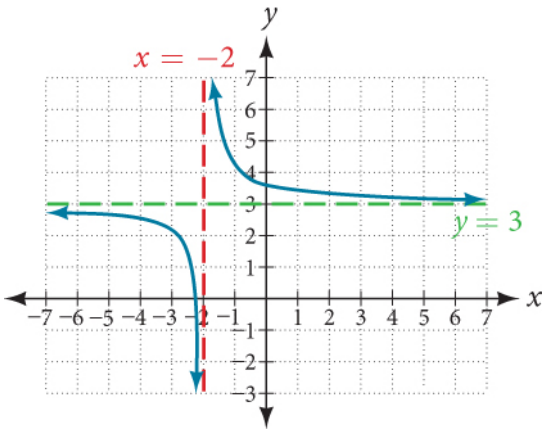


Figure 7

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^-,\ f(x) \to -\infty, \text{and as } x \to -2^+,\ f(x) \to \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty,\ f(x) \to 3.$$

Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

TRY IT #2 Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solving Applied Problems Involving Rational Functions

In Example 2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Rational Function

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \ldots + b_1 x + b_0}, Q(x) \neq 0$$

EXAMPLE 3

Solving an Applied Problem Involving a Rational Function

After running out of pre-packaged supplies, a nurse in a refugee camp is preparing an intravenous sugar solution for patients in the camp hospital. A large mixing tank currently contains 100 gallons of distilled water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of distilled water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the ratio of sugar to water, in pounds per gallon in the tank after 12 minutes. Is that a greater ratio of sugar to water, in pounds per gallon than at the beginning?

Solution

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{aligned} \text{water: } W(t) &= 100 + 10t \text{ in gallons} \\ \text{sugar: } S(t) &= 5 + 1t \text{ in pounds} \end{aligned}$$

The ratio of sugar to water, in pounds per gallon, C, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5 + t}{100 + 10t}$$

The ratio of sugar to water, in pounds per gallon after 12 minutes is given by evaluating $C(t)$ at $t = 12$.

$$\begin{aligned} C(12) &= \frac{5+12}{100+10(12)} \\ &= \frac{17}{220} \end{aligned}$$

This means the ratio of sugar to water, in pounds per gallon is 17 pounds of sugar to 220 gallons of water.

At the beginning, the ratio of sugar to water, in pounds per gallon is

$$\begin{aligned} C(0) &= \frac{5+0}{100+10(0)} \\ &= \frac{1}{20} \end{aligned}$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$, the ratio of sugar to water, in pounds per gallon is greater after 12 minutes than at the beginning.

TRY IT #3 There are 1,200 first-year and 1,500 second-year students at a rally at noon. After 12 p.m., 20 first-year students arrive at the rally every five minutes while 15 second-year students leave the rally. Find the ratio of first-year to second-year students at 1 p.m.

Finding the Domains of Rational Functions

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Domain of a Rational Function

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

HOW TO

Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

EXAMPLE 4

Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure 8, confirms that the function is not defined when $x = \pm 3$.

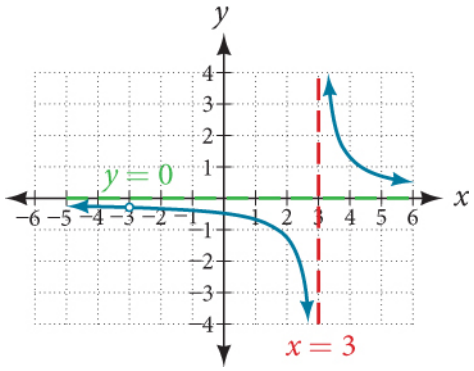


Figure 8

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

TRY IT #4 Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

HOW TO

Given a rational function, identify any vertical asymptotes of its graph.

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

EXAMPLE 5

Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$\begin{aligned} k(x) &= \frac{5+2x^2}{2-x-x^2} \\ &= \frac{5+2x^2}{(2+x)(1-x)} \end{aligned}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} (2+x)(1-x) &= 0 \\ x &= -2, 1 \end{aligned}$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 9 confirms the location of the two vertical asymptotes.

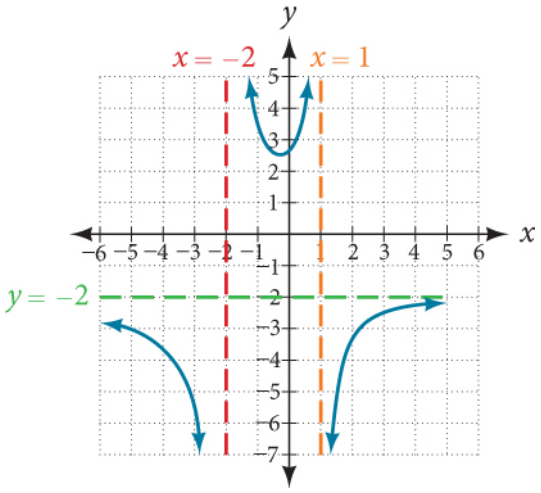


Figure 9

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a **removable discontinuity**.

For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that $x-3$ is not a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See Figure 10. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

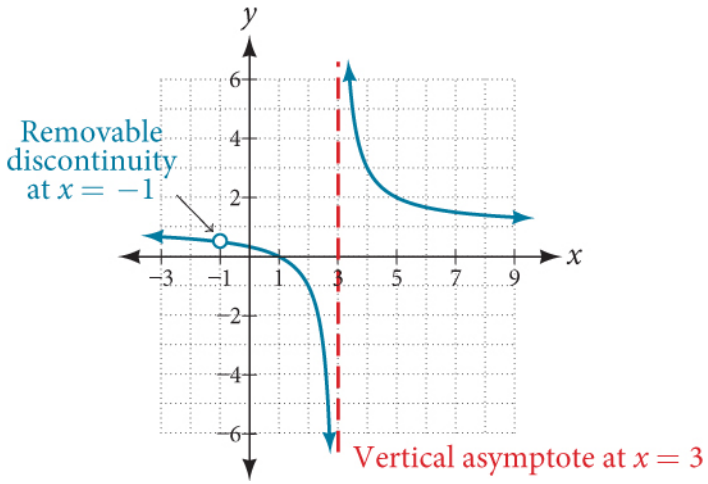


Figure 10

Removable Discontinuities of Rational Functions

A **removable discontinuity** occurs in the graph of a rational function at $x=a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

EXAMPLE 6

Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure 11.

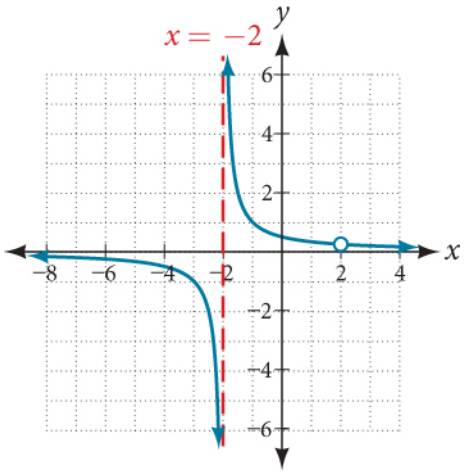


Figure 11

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

TRY IT #5 Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.

$$\text{Example: } f(x) = \frac{4x+2}{x^2+4x-5}$$

In this case, the end behavior is $f(x) \approx \frac{4x}{x^2} = \frac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See Figure 12. Note that this graph crosses the horizontal asymptote.

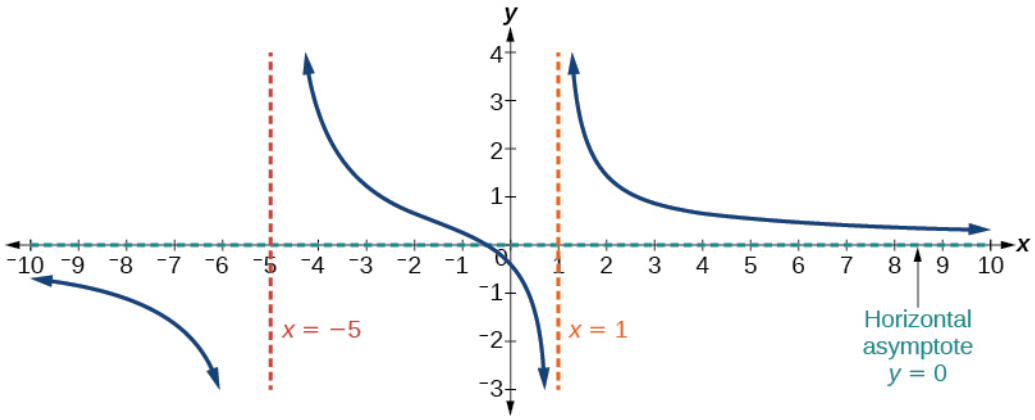


Figure 12 Horizontal asymptote $y = 0$ when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p <$ degree of q.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

$$\text{Example: } f(x) = \frac{3x^2 - 2x + 1}{x - 1}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\frac{3x^2-2x+1}{x-1}$. The quotient is $3x + 1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See Figure 13.

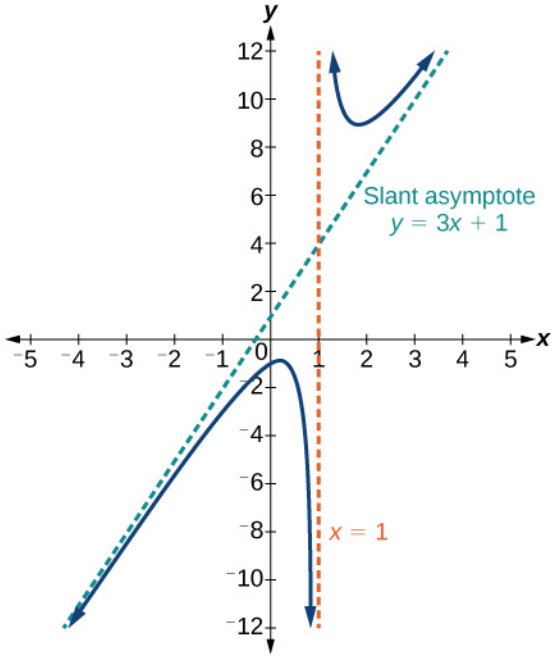


Figure 13 Slant asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p >$ degree of q by 1.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$.

$$\text{Example: } f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 14. Note that this graph crosses the horizontal asymptote.

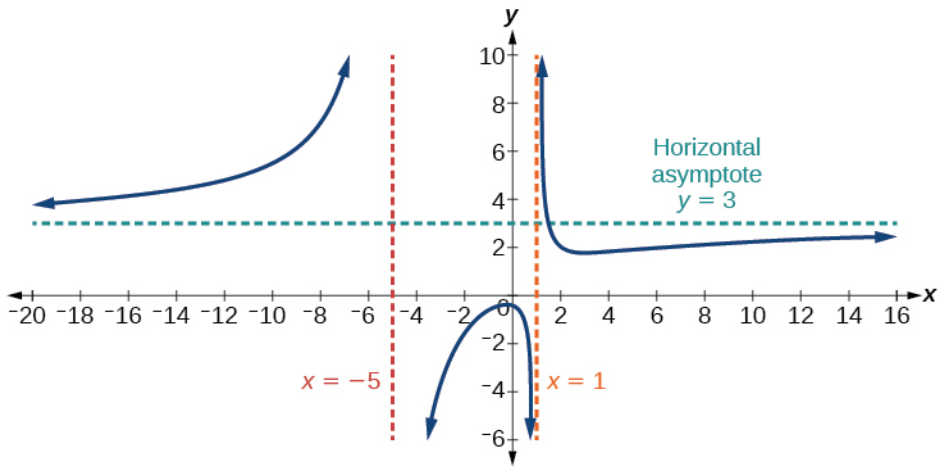


Figure 14 Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p = degree of q.

Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4,$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty,\ f(x) \to \infty$$

Horizontal Asymptotes of Rational Functions

The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator *is less than* degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator *is greater than degree of denominator by one*: no horizontal asymptote; slant asymptote.
- Degree of numerator *is equal to* degree of denominator: horizontal asymptote at ratio of leading coefficients.

EXAMPLE 7

Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

ⓐ $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$ ⓑ $h(x) = \frac{x^2 - 4x + 1}{x + 2}$ ⓒ $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

ⓐ $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of p = degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.

ⓑ $h(x) = \frac{x^2-4x+1}{x+2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\frac{x^2-4x+1}{x+2}$.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

The quotient is $x-6$ and the remainder is 13. There is a slant asymptote at $y = x-6$.

ⓒ $k(x) = \frac{x^2+4x}{x^3-8}$: The degree of $p = 2$ $<$ degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

EXAMPLE 8

Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \frac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \frac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

EXAMPLE 9

Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2, \text{and } 5$, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty,\ f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 15.

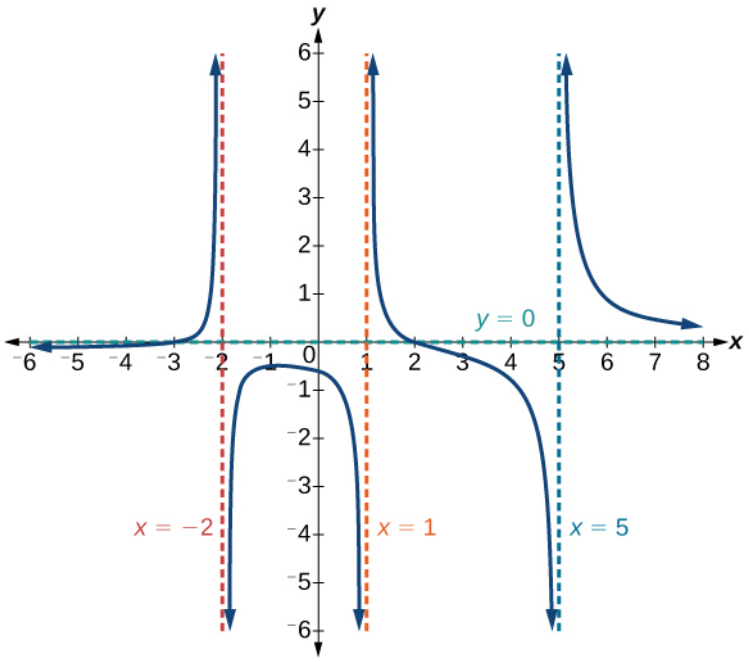


Figure 15

TRY IT #6 Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Intercepts of Rational Functions

A rational function will have a *y*-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a *y*-intercept if the function is not defined at zero.

Likewise, a rational function will have *x*-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, *x*-intercepts can only occur when the numerator of the rational function is equal to zero.

EXAMPLE 10

Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the *y*-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= \frac{-6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The *x*-intercepts will occur when the function is equal to zero:

$$\begin{aligned} 0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} && \text{This is zero when the numerator is zero.} \\ 0 &= (x-2)(x+3) \\ x &= 2, -3 \end{aligned}$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 16.

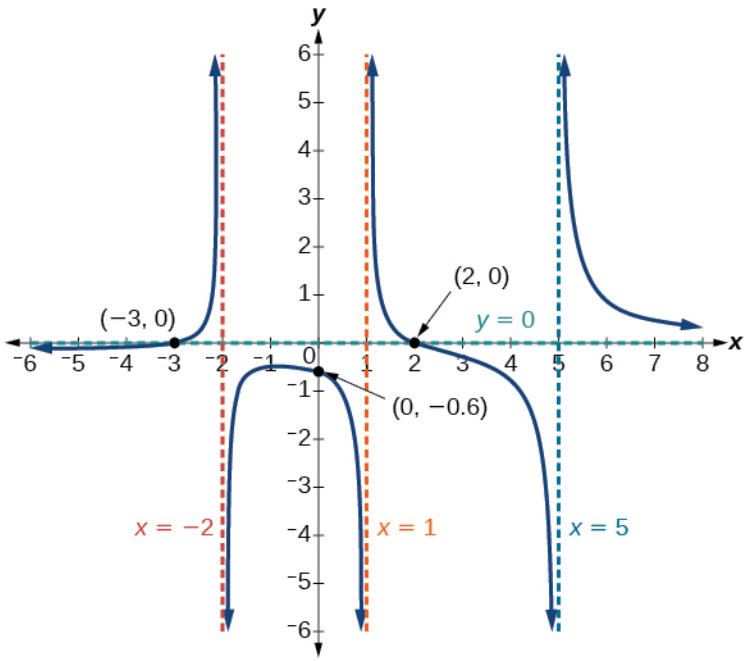


Figure 16

TRY IT #7 Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Graphing Rational Functions

In Example 9, we see that the numerator of a rational function reveals the x-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 17.

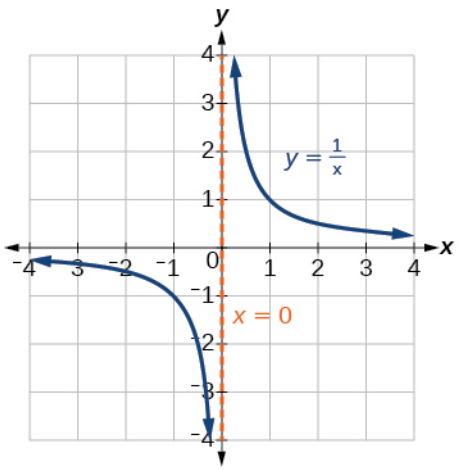


Figure 17

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 18.

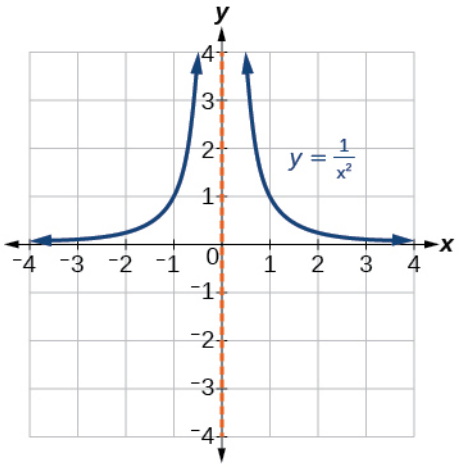


Figure 18

For example, the graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 19.

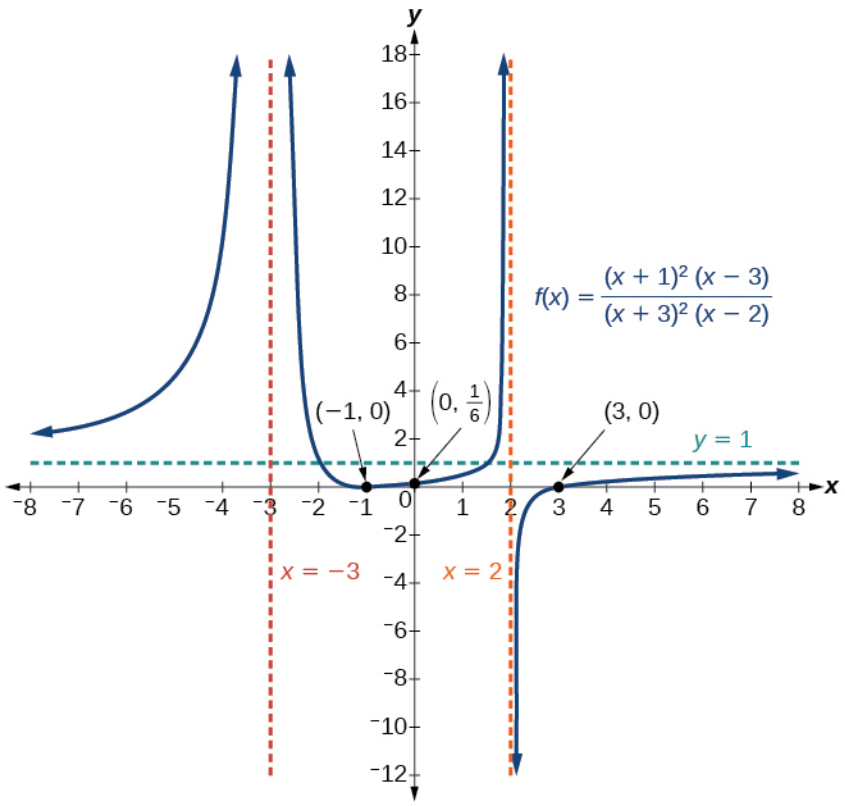


Figure 19

- At the *x*-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the *x*-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side.

HOW TO

Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the *y*-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the *x*-intercepts.
4. Find the multiplicities of the *x*-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

EXAMPLE 11

Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the *y*-intercept:

$$\begin{aligned} f(0) &= \frac{(0+2)(0-3)}{(0+1)^2(0-2)} \\ &= 3 \end{aligned}$$

To find the *x*-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find *x*-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a *y*-intercept at $(0, 3)$ and *x*-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no *x*-intercepts between the vertical asymptotes, and the *y*-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 20.

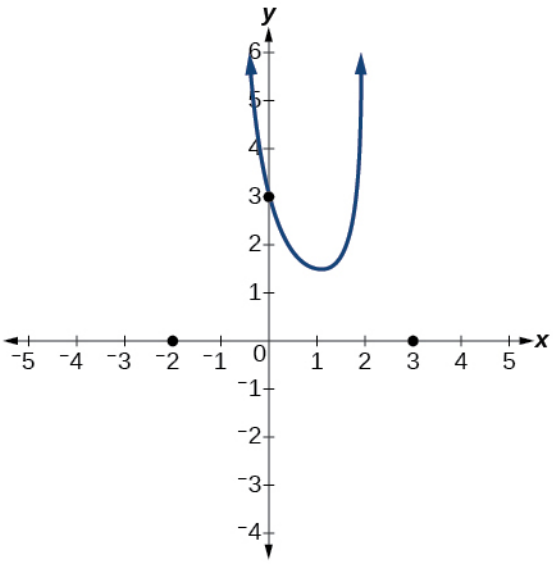


Figure 20

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 21. After passing through the *x*-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

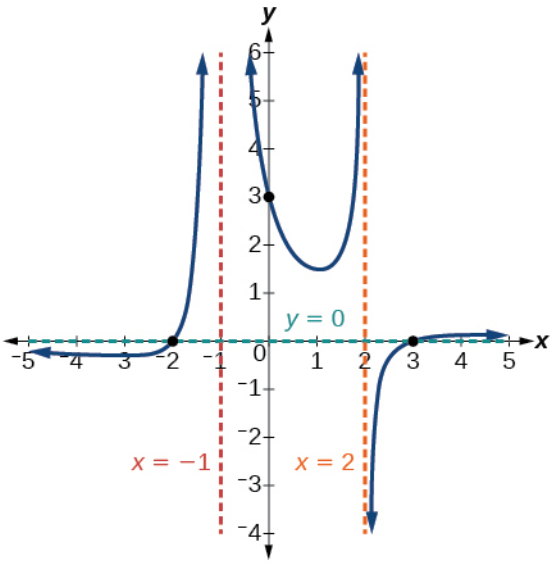


Figure 21

TRY IT #8 Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has *x*-intercepts at $x = x_1, x_2, ..., x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i = \text{any } v_j$, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the *x*-intercept or by the horizontal asymptote if it is nonzero.

HOW TO

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to

determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

EXAMPLE 12

Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 22.

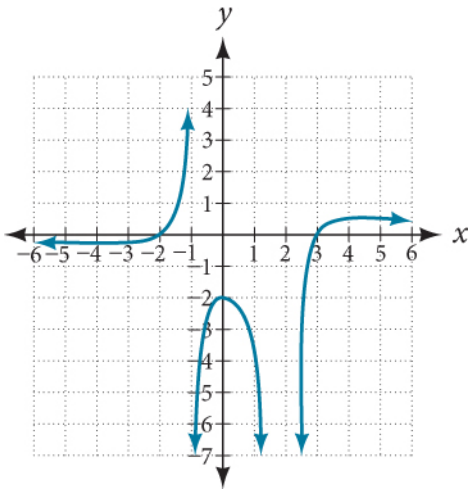


Figure 22

Solution

The graph appears to have *x*-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 23.

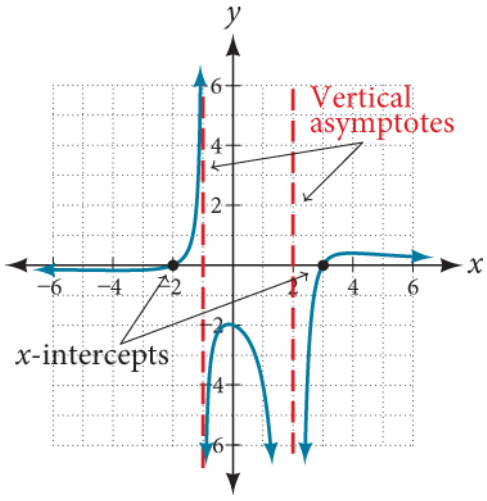


Figure 23

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0, -2)$.

$$\begin{aligned} -2 &= a\frac{(0+2)(0-3)}{(0+1)(0-2)^2} \\ -2 &= a\frac{-6}{4} \\ a &= \frac{-8}{-6} = \frac{4}{3} \end{aligned}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

MEDIA

Access these online resources for additional instruction and practice with rational functions.

Graphing Rational Functions (https://openstax.org/l/graphrational)
Find the Equation of a Rational Function (https://openstax.org/l/equatrational)
Determining Vertical and Horizontal Asymptotes (https://openstax.org/l/asymptote)
Find the Intercepts, Asymptotes, and Hole of a Rational Function (https://openstax.org/l/interasymptote)

5.6 SECTION EXERCISES

Verbal

1. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

2. What is the fundamental difference in the graphs of polynomial functions and rational functions?

3. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

4. Can a graph of a rational function have no vertical asymptote? If so, how?

5. Can a graph of a rational function have no x-intercepts? If so, how?

Algebraic

For the following exercises, find the domain of the rational functions.

6. $f(x) = \frac{x-1}{x+2}$

7. $f(x) = \frac{x+1}{x^2-1}$

8. $f(x) = \frac{x^2+4}{x^2-2x-8}$

9. $f(x) = \frac{x^2+4x-3}{x^4-5x^2+4}$

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

10. $f(x) = \frac{4}{x-1}$

11. $f(x) = \frac{2}{5x+2}$

12. $f(x) = \frac{x}{x^2-9}$

13. $f(x) = \frac{x}{x^2+5x-36}$

14. $f(x) = \frac{3+x}{x^3-27}$

15. $f(x) = \frac{3x-4}{x^3-16x}$

16. $f(x) = \frac{x^2-1}{x^3+9x^2+14x}$

17. $f(x) = \frac{x+5}{x^2-25}$

18. $f(x) = \frac{x-4}{x-6}$

19. $f(x) = \frac{4-2x}{3x-1}$

For the following exercises, find the x- and y-intercepts for the functions.

20. $f(x) = \frac{x+5}{x^2+4}$

21. $f(x) = \frac{x}{x^2-x}$

22. $f(x) = \frac{x^2+8x+7}{x^2+11x+30}$

23. $f(x) = \frac{x^2+x+6}{x^2-10x+24}$

24. $f(x) = \frac{94-2x^2}{3x^2-12}$

For the following exercises, describe the local and end behavior of the functions.

25. $f(x)=\frac{x}{2x+1}$

26. $f(x)=\frac{2x}{x-6}$

27. $f(x)=\frac{-2x}{x-6}$

28. $f(x)=\frac{x^2-4x+3}{x^2-4x-5}$

29. $f(x)=\frac{2x^2-32}{6x^2+13x-5}$

For the following exercises, find the slant asymptote of the functions.

30. $f(x)=\frac{24x^2+6x}{2x+1}$

31. $f(x)=\frac{4x^2-10}{2x-4}$

32. $f(x)=\frac{81x^2-18}{3x-2}$

33. $f(x)=\frac{6x^3-5x}{3x^2+4}$

34. $f(x)=\frac{x^2+5x+4}{x-1}$

Graphical

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

35. The reciprocal function shifted up two units.

36. The reciprocal function shifted down one unit and left three units.

37. The reciprocal squared function shifted to the right 2 units.

38. The reciprocal squared function shifted down 2 units and right 1 unit.

For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

39. $p(x)=\frac{2x-3}{x+4}$

40. $q(x)=\frac{x-5}{3x-1}$

41. $s(x)=\frac{4}{(x-2)^2}$

42. $r(x)=\frac{5}{(x+1)^2}$

43. $f(x)=\frac{3x^2-14x-5}{3x^2+8x-16}$

44. $g(x)=\frac{2x^2+7x-15}{3x^2-14x+15}$

45. $a(x)=\frac{x^2+2x-3}{x^2-1}$

46. $b(x)=\frac{x^2-x-6}{x^2-4}$

47. $h(x)=\frac{2x^2+x-1}{x-4}$

48. $k(x)=\frac{2x^2-3x-20}{x-5}$

49. $w(x)=\frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

50. $z(x)=\frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

For the following exercises, write an equation for a rational function with the given characteristics.

51. Vertical asymptotes at $x=5$ and $x=-5$, x-intercepts at $(2,0)$ and $(-1,0)$, y-intercept at $(0,4)$

52. Vertical asymptotes at $x=-4$ and $x=-1$, x-intercepts at $(1,0)$ and $(5,0)$, y-intercept at $(0,7)$

53. Vertical asymptotes at $x=-4$ and $x=-5$, x-intercepts at $(4,0)$ and $(-6,0)$, Horizontal asymptote at $y=7$

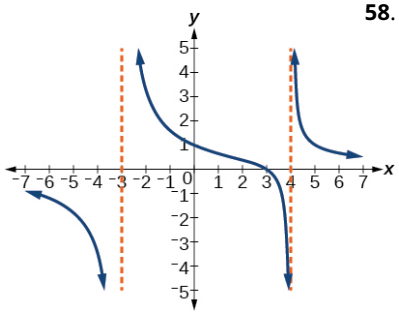
54. Vertical asymptotes at $x=-3$ and $x=6$, x-intercepts at $(-2,0)$ and $(1,0)$, Horizontal asymptote at $y=-2$

55. Vertical asymptote at $x=-1$, Double zero at $x=2$, y-intercept at $(0,2)$

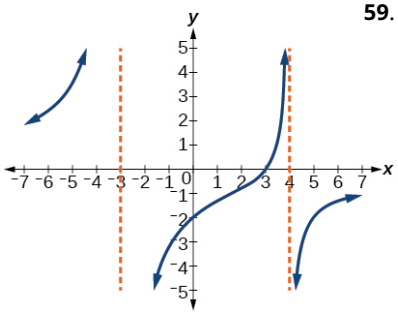
56. Vertical asymptote at $x=3$, Double zero at $x=1$, y-intercept at $(0,4)$

For the following exercises, use the graphs to write an equation for the function.

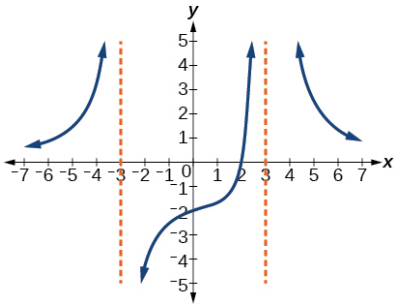
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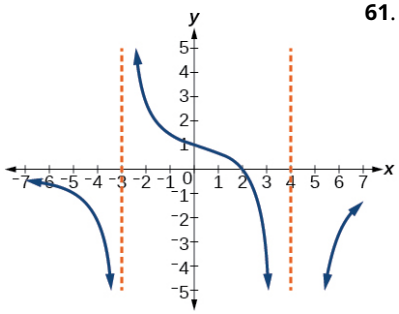
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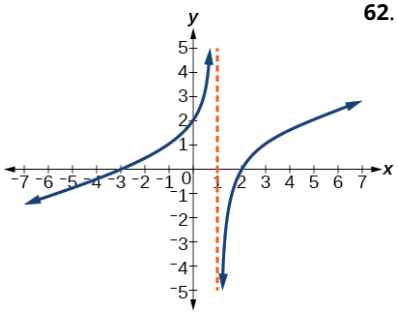
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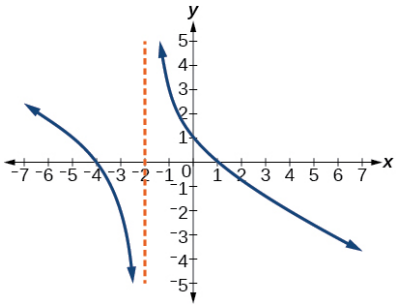
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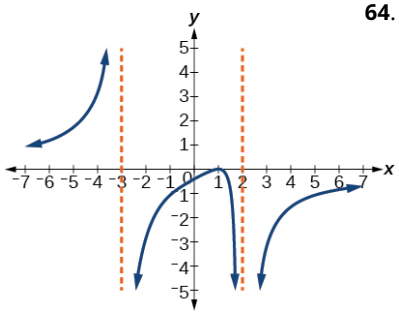
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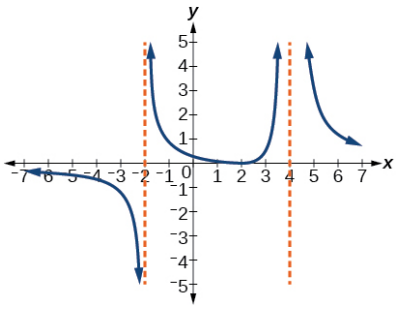
62.



63.



64. Use $\left(0,-\frac{1}{2}\right)$ as the additional point.



Numeric

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

65. $f(x)=\frac{1}{x-2}$

66. $f(x)=\frac{x}{x-3}$

67. $f(x)=\frac{2x}{x+4}$

68. $f(x)=\frac{2x}{(x-3)^2}$

69. $f(x)=\frac{x^2}{x^2+2x+1}$

Technology

For the following exercises, use a calculator to graph $f(x)$. Use the graph to solve $f(x)>0$.

70. $f(x)=\frac{2}{x+1}$

71. $f(x)=\frac{4}{2x-3}$

72. $f(x)=\frac{2}{(x-1)(x+2)}$

73. $f(x)=\frac{x+2}{(x-1)(x-4)}$

74. $f(x)=\frac{(x+3)^2}{(x-1)^2(x+1)}$

Extensions

For the following exercises, identify the removable discontinuity.

75. $f(x) = \frac{x^2-4}{x-2}$

76. $f(x) = \frac{x^3+1}{x+1}$

77. $f(x) = \frac{x^2+x-6}{x-2}$

78. $f(x) = \frac{2x^2+5x-3}{x+3}$

79. $f(x) = \frac{x^3+x^2}{x+1}$

Real-World Applications

For the following exercises, express a rational function that describes the situation.

80. In the refugee camp hospital, a large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

81. In the refugee camp hospital, a large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

For the following exercises, use the given rational function to answer the question.

82. The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C(t) = \frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?

83. The concentration C of a drug in a patient's bloodstream $_t$ hours after injection is given by $C(t) = \frac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

84. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.

85. A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.

86. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.

87. A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.

88. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

5.7 Inverses and Radical Functions

Learning Objectives

In this section, you will:

- Find the inverse of an invertible polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

Park rangers and other trail managers may construct rock piles, stacks, or other arrangements, usually called cairns, to mark trails or other landmarks. (Rangers and environmental scientists discourage hikers from doing the same, in order to avoid confusion and preserve the habitats of plants and animals.) A cairn in the form of a mound of gravel is in the shape of a cone with the height equal to twice the radius.



Figure 1

The volume is found using a formula from elementary geometry.

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{3}\pi r^2 (2r) \\ &= \tfrac{2}{3}\pi r^3 \end{aligned}$$

We have written the volume V in terms of the radius r. However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This function is the inverse of the formula for V in terms of r.

In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

Finding the Inverse of a Polynomial Function

Two functions f and g are inverse functions if for every coordinate pair in $f, (a, b)$, there exists a corresponding coordinate pair in the inverse function, $g, (b, \ a)$. In other words, the coordinate pairs of the inverse functions have the input and output interchanged. Only one-to-one functions have inverses. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For example, suppose the Sustainability Club builds a water runoff collector in the shape of a parabolic trough as shown in Figure 2. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.

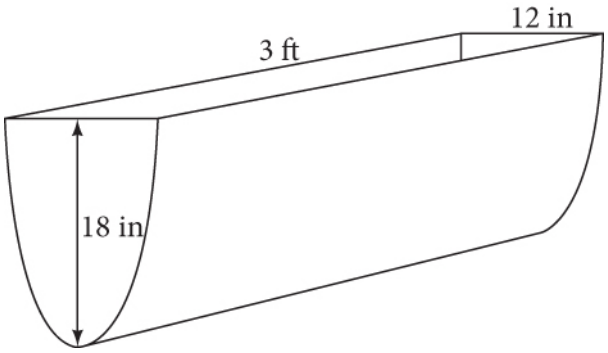


Figure 2

Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola. See Figure 3.

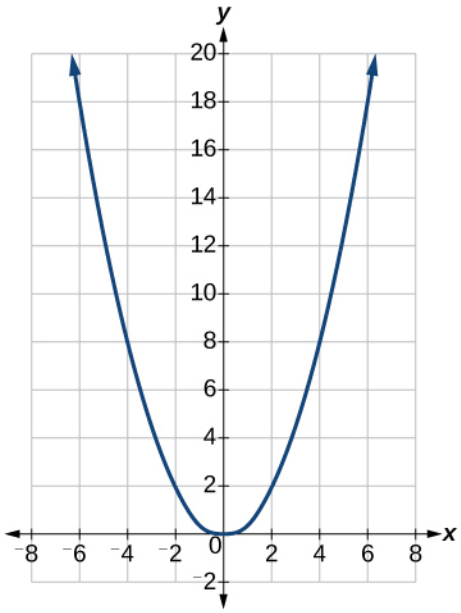


Figure 3

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point (6, 18), from which we can solve for the stretch factor a.

$$\begin{aligned} 18 &= a6^2 \\ a &= \frac{18}{36} \\ &= \frac{1}{2} \end{aligned}$$

Our parabolic cross section has the equation

$$y(x) = \frac{1}{2}x^2$$

We are interested in the surface area of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth y, the width will be given by $2x$, so we need to solve the equation above for x and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$\begin{aligned} y &= \tfrac{1}{2}x^2 \\ 2y &= x^2 \\ x &= \pm\sqrt{2y} \\ \therefore y &= \pm\sqrt{2x} \end{aligned}$$

This is not a function as written. Since we are limiting ourselves to positive x values in the original function, we can eliminate the negative solution, which gives us the inverse function we're looking for.

$$y = \sqrt{2x}$$

Because x is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:

$$\begin{aligned} \text{Area} &= l \cdot w \\ &= 36 \cdot 2x \\ &= 72x \\ &= 72\sqrt{2y} \end{aligned}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called radical functions. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write $(f(x))^{-1} = \frac{1}{f(x)}$.

An important relationship between inverse functions is that they "undo" each other. If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. In other words, whatever the function f does to x, f^{-1} undoes it—and vice-versa.

$$f^{-1}(f(x)) = x, \quad \text{for all } x \text{ in the domain of } f$$

and

$$f\left(f^{-1}(x)\right) = x, \quad \text{for all } x \text{ in the domain of } f^{-1}$$

Note that the inverse switches the domain and range of the original function.

Verifying Two Functions Are Inverses of One Another

Two functions, f and g, are inverses of one another if for all x in the domain of f and g.

$$g(f(x)) = f(g(x)) = x$$



HOW TO

Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one.

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y, and rename the function $f^{-1}(x)$.

EXAMPLE 1

Verifying Inverse Functions

Show that $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses, for $x \neq 0, -1$.

Solution

We must show that $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$.

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\tfrac{1}{x+1}\right) \\ &= \frac{1}{\frac{1}{x+1}} - 1 \\ &= (x+1) - 1 \\ &= x \\ f(f^{-1}(x)) &= f\left(\tfrac{1}{x} - 1\right) \\ &= \frac{1}{\left(\frac{1}{x} - 1\right) + 1} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

Therefore, $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses.

TRY IT #1 Show that $f(x) = \frac{x+5}{3}$ and $f^{-1}(x) = 3x - 5$ are inverses.

EXAMPLE 2

Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

Solution

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

$$\begin{aligned} y &= 5x^3 + 1 \\ x &= 5y^3 + 1 \\ x - 1 &= 5y^3 \\ \tfrac{x-1}{5} &= y^3 \\ f^{-1}(x) &= \sqrt[3]{\tfrac{x-1}{5}} \end{aligned}$$

Analysis

Look at the graph of f and f^{-1}. Notice that one graph is the reflection of the other about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging x and y, notice corresponding points. If (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}. Since $(0, 1)$ is on the graph of f, then $(1, 0)$ is on the graph of f^{-1}. Similarly, since $(1, 6)$ is on the graph of f, then $(6, 1)$ is on the graph of f^{-1}. See Figure 4.

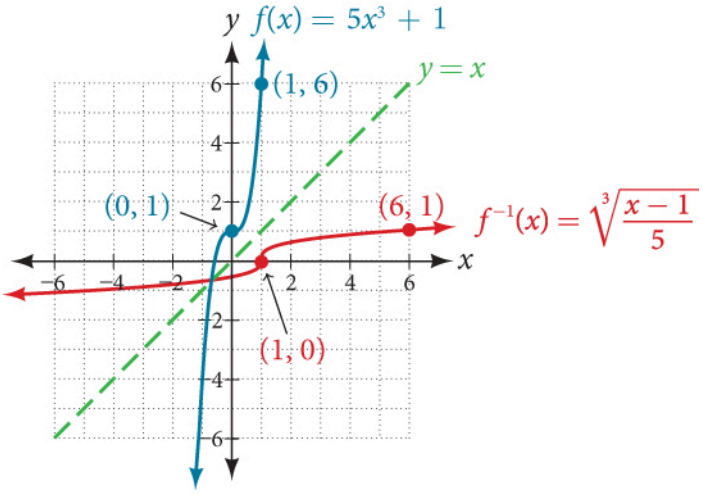


Figure 4

TRY IT #2 Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

Restricting the Domain

If a function is not one-to-one, it cannot have an inverse. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.

HOW TO

Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse.

1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace $f(x)$ with y.
3. Interchange x and y.
4. Solve for y, and rename the function or pair of function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

EXAMPLE 3

Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f:

(a) $f(x) = (x-4)^2,\ x \geq 4$ (b) $f(x) = (x-4)^2,\ x \leq 4$

Solution

The original function $f(x) = (x-4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one. See Figure 5.

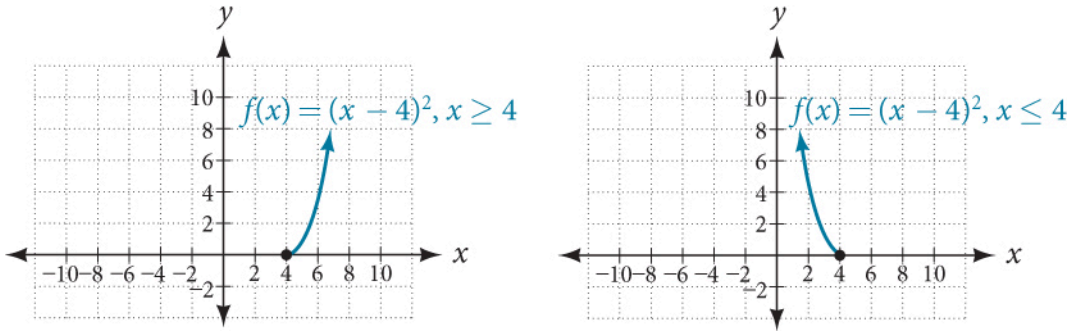


Figure 5

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$\begin{aligned} y &= (x-4)^2 && \text{Interchange } x \text{ and } y. \\ x &= (y-4)^2 && \text{Take the square root.} \\ \pm\sqrt{x} &= y-4 && \text{Add 4 to both sides.} \\ 4 \pm \sqrt{x} &= y \end{aligned}$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of x and y for the original $f(x)$, we looked at the domain: the values x could assume. When we reversed the roles of x and y, this gave us the values y could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

(a) The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the + case:

$$f^{-1}(x) = 4 + \sqrt{x}$$

(b) The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the – case:

$$f^{-1}(x) = 4 - \sqrt{x}$$

Analysis

On the graphs in Figure 6, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4,\ 0)$ is on the graph of f and the coordinate pair $(0,\ 4)$ is on the graph of f^{-1}. For any coordinate pair, if $(a,\ b)$ is on the graph of f, then $(b,\ a)$ is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y = x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y = x$.

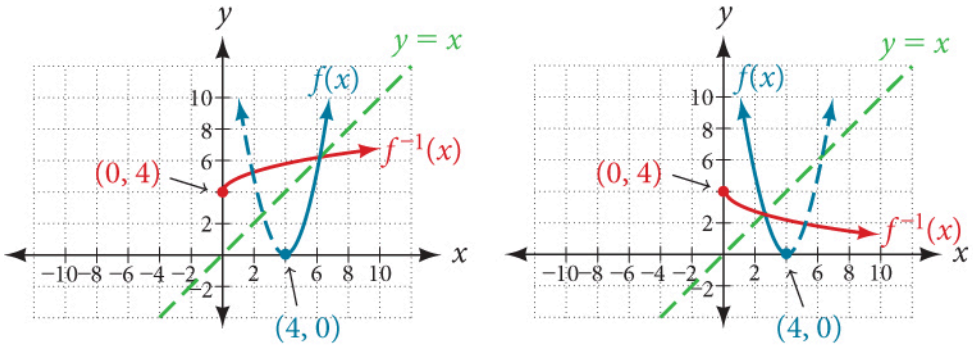


Figure 6

EXAMPLE 4

Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

Solution

We can see this is a parabola with vertex at $(2, -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing $f(x)$ with a simple variable, y, then solve for x.

$$\begin{aligned}
y &= (x-2)^2 - 3 && \text{Interchange } x \text{ and } y. \\
x &= (y-2)^2 - 3 && \text{Add 3 to both sides.} \\
x+3 &= (y-2)^2 && \text{Take the square root.} \\
\pm\sqrt{x+3} &= y-2 && \text{Add 2 to both sides.} \\
2 \pm \sqrt{x+3} &= y && \text{Rename the function.} \\
f^{-1}(x) &= 2 \pm \sqrt{x+3}
\end{aligned}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the + case

$$f^{-1}(x) = 2 + \sqrt{x+3}$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just have easily opted to restrict the domain on $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 7. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2, \quad -3)$ is on the graph of f and the coordinate pair $(-3, \quad 2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = \left[2, \infty\right)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = \left[-3, \infty\right).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.

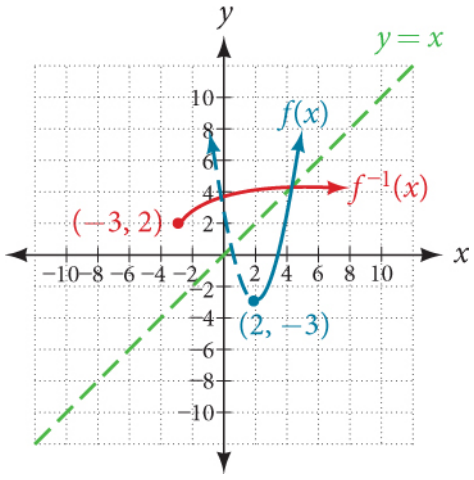


Figure 7

TRY IT #3 Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Solving Applications of Radical Functions

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

HOW TO

Given a radical function, find the inverse.

1. Determine the range of the original function.
2. Replace $f(x)$ with y, then solve for x.
3. If necessary, restrict the domain of the inverse function to the range of the original function.

EXAMPLE 5

Finding the Inverse of a Radical Function

Restrict the domain of the function $f(x) = \sqrt{x-4}$ and then find the inverse.

Solution

Note that the original function has range $f(x) \geq 0$. Replace $f(x)$ with y, then solve for x.

$$\begin{aligned} y &= \sqrt{x-4} && \text{Replace } f(x) \text{ with } y. \\ x &= \sqrt{y-4} && \text{Interchange } x \text{ and } y. \\ x &= \sqrt{y-4} && \text{Square each side.} \\ x^2 &= y-4 && \text{Add 4.} \\ x^2+4 &= y && \text{Rename the function } f^{-1}(x). \\ f^{-1}(x) &= x^2+4 \end{aligned}$$

Recall that the domain of this function must be limited to the range of the original function.

$$f^{-1}(x) = x^2 + 4, x \geq 0$$

Analysis

Notice in Figure 8 that the inverse is a reflection of the original function over the line $y = x$. Because the original function

has only positive outputs, the inverse function has only positive inputs.

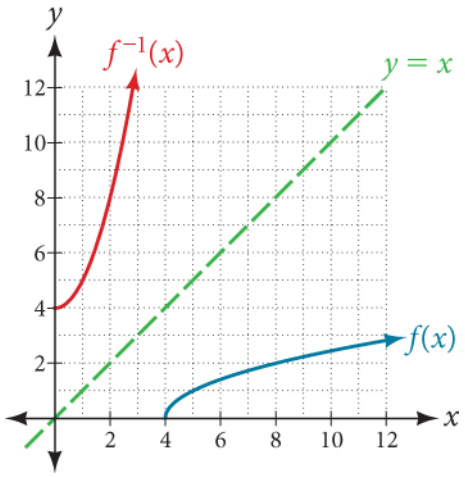


Figure 8

TRY IT #4 Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x+3}$.

Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

EXAMPLE 6

Solving an Application with a Cubic Function

Park rangers construct a mound of gravel in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius r. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

Solution

Start with the given function for V. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of 0. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for r in terms of V, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

$$\begin{aligned} V &= \tfrac{2}{3}\pi r^3 \\ r^3 &= \tfrac{3V}{2\pi} && \text{Solve for } r^3. \\ r &= \sqrt[3]{\tfrac{3V}{2\pi}} && \text{Solve for } r. \end{aligned}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi = 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\tfrac{3V}{2\pi}} \\ &= \sqrt[3]{\tfrac{3 \cdot 100}{2 \cdot 3.14}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

EXAMPLE 7

Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function $f(x) = \sqrt{\frac{(x+2)(x-3)}{(x-1)}}$.

Solution

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(x+2)(x-3)}{(x-1)} \geq 0$. The output of a rational function can change signs (change from positive to negative or vice versa) at x-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x = -2,\ 1,\ \text{and}\ 3$.

To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph as shown in Figure 9.

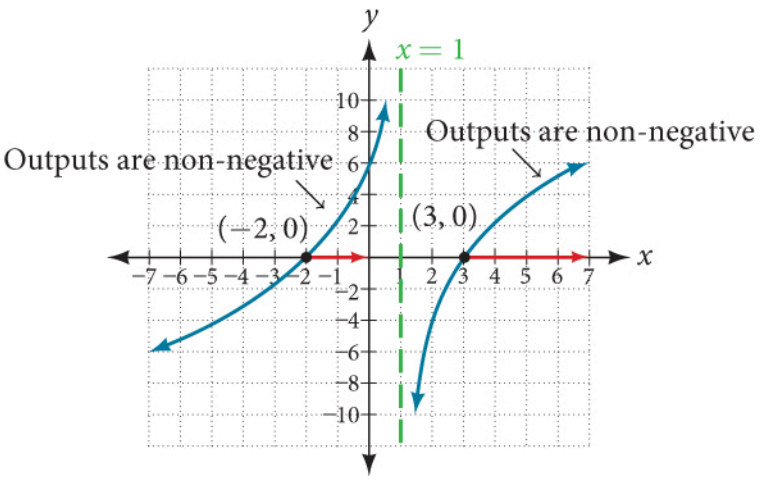


Figure 9

This function has two x-intercepts, both of which exhibit linear behavior near the x-intercepts. There is one vertical asymptote, corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. There is a y-intercept at $(0, \sqrt{6})$.

From the y-intercept and x-intercept at $x = -2$, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2, 1) \cup [3, \infty)$.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

EXAMPLE 8

Finding the Inverse of a Rational Function

The function $C = \frac{20+0.4n}{100+n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then

use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution

We first want the inverse of the function in order to determine how many mL we need for a given concentration. We will solve for n in terms of C.

$$\begin{aligned} C &= \frac{20+0.4n}{100+n} \\ C(100+n) &= 20+0.4n \\ 100C+Cn &= 20+0.4n \\ 100C-20 &= 0.4n-Cn \\ 100C-20 &= (0.4-C)n \\ n &= \frac{100C-20}{0.4-C} \end{aligned}$$

Now evaluate this function at 35%, which is $C = 0.35$.

$$\begin{aligned} n &= \frac{100(0.35)-20}{0.4-0.35} \\ &= \frac{15}{0.05} \\ &= 300 \end{aligned}$$

We can conclude that 300 mL of the 40% solution should be added.

TRY IT #5 Find the inverse of the function $f(x) = \frac{x+3}{x-2}$.

MEDIA

Access these online resources for additional instruction and practice with inverses and radical functions.

Graphing the Basic Square Root Function (https://openstax.org/l/graphsquareroot)
Find the Inverse of a Square Root Function (https://openstax.org/l/inversesquare)
Find the Inverse of a Rational Function (https://openstax.org/l/inverserational)
Find the Inverse of a Rational Function and an Inverse Function Value (https://openstax.org/l/rationalinverse)
Inverse Functions (https://openstax.org/l/inversefunction)

5.7 SECTION EXERCISES

Verbal

1. Explain why we cannot find inverse functions for all polynomial functions.

2. Why must we restrict the domain of a quadratic function when finding its inverse?

3. When finding the inverse of a radical function, what restriction will we need to make?

4. The inverse of a quadratic function will always take what form?

Algebraic

For the following exercises, find the inverse of the function on the given domain.

5. $f(x) = (x-4)^2,\ [4,\infty)$

6. $f(x) = (x+2)^2,\ [-2,\infty)$

7. $f(x) = (x+1)^2 - 3,\ [-1,\infty)$

8. $f(x) = 3x^2 + 5,\ \left(\infty, 0\right]$

9. $f(x) = 12 - x^2,\ [0,\infty)$

10. $f(x) = 9 - x^2,\ [0,\infty)$

11. $f(x) = 2x^2 + 4,\ [0,\infty)$

For the following exercises, find the inverse of the functions.

12. $f(x) = x^3 + 5$ **13.** $f(x) = 3x^3 + 1$ **14.** $f(x) = 4 - x^3$

15. $f(x) = 4 - 2x^3$

For the following exercises, find the inverse of the functions.

16. $f(x) = \sqrt{2x+1}$ **17.** $f(x) = \sqrt{3-4x}$ **18.** $f(x) = 9 + \sqrt{4x-4}$

19. $f(x) = \sqrt{6x-8} + 5$ **20.** $f(x) = 9 + 2\sqrt[3]{x}$ **21.** $f(x) = 3 - \sqrt[3]{x}$

22. $f(x) = \frac{2}{x+8}$ **23.** $f(x) = \frac{3}{x-4}$ **24.** $f(x) = \frac{x+3}{x+7}$

25. $f(x) = \frac{x-2}{x+7}$ **26.** $f(x) = \frac{3x+4}{5-4x}$ **27.** $f(x) = \frac{5x+1}{2-5x}$

28. $f(x) = x^2 + 2x,\ [-1, \infty)$ **29.** $f(x) = x^2 + 4x + 1,\ [-2, \infty)$ **30.** $f(x) = x^2 - 6x + 3,\ [3, \infty)$

Graphical

For the following exercises, find the inverse of the function and graph both the function and its inverse.

31. $f(x) = x^2 + 2,\ \ x \geq 0$ **32.** $f(x) = 4 - x^2,\ \ x \geq 0$ **33.** $f(x) = (x+3)^2,\ \ x \geq -3$

34. $f(x) = (x-4)^2,\ \ x \geq 4$ **35.** $f(x) = x^3 + 3$ **36.** $f(x) = 1 - x^3$

37. $f(x) = x^2 + 4x,\ \ x \geq -2$ **38.** $f(x) = x^2 - 6x + 1,\ \ x \geq 3$ **39.** $f(x) = \frac{2}{x}$

40. $f(x) = \frac{1}{x^2},\ \ x \geq 0$

For the following exercises, use a graph to help determine the domain of the functions.

41. $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$ **42.** $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$ **43.** $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$

44. $f(x) = \sqrt{\frac{x^2-x-20}{x-2}}$ **45.** $f(x) = \sqrt{\frac{9-x^2}{x+4}}$

Technology

For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with y-coordinates given.

46. $f(x) = x^3 - x - 2, y = 1, 2, 3$ **47.** $f(x) = x^3 + x - 2, y = 0, 1, 2$ **48.** $f(x) = x^3 + 3x - 4, y = 0, 1, 2$

49. $f(x) = x^3 + 8x - 4, y = -1, 0, 1$ **50.** $f(x) = x^4 + 5x + 1, y = -1, 0, 1$

Extensions

For the following exercises, find the inverse of the functions with a, b, c positive real numbers.

51. $f(x) = ax^3 + b$ **52.** $f(x) = x^2 + bx$ **53.** $f(x) = \sqrt{ax^2 - b}$

54. $f(x) = \sqrt[3]{ax + b}$ **55.** $f(x) = \frac{ax+b}{x+c}$

Real-World Applications

For the following exercises, determine the function described and then use it to answer the question.

56. An object dropped from a height of 200 meters has a height, $h(t)$, in meters after t seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express t as a function of height, h, and find the time to reach a height of 50 meters.

57. An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.

58. The volume, V, of a sphere in terms of its radius, r, is given by $V(r) = \frac{4}{3}\pi r^3$. Express r as a function of V, and find the radius of a sphere with volume of 200 cubic feet.

59. The surface area, A, of a sphere in terms of its radius, r, is given by $A(r) = 4\pi r^2$. Express r as a function of A, and find the radius of a sphere with a surface area of 1000 square inches.

60. A container holds 100 mL of a solution that is 25 mL acid. If n mL of a solution that is 60% acid is added, the function $C(n) = \frac{25+.6n}{100+n}$ gives the concentration, C, as a function of the number of mL added, n. Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.

61. The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.

62. The volume of a cylinder , V, in terms of radius, r, and height, h, is given by $V = \pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.

63. The surface area, A, of a cylinder in terms of its radius, r, and height, h, is given by $A = 2\pi r^2 + 2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of A and find the radius if the surface area is 200 square feet.

64. The volume of a right circular cone, V, in terms of its radius, r, and its height, h, is given by $V = \frac{1}{3}\pi r^2 h$. Express r in terms of V if the height of the cone is 12 feet and find the radius of a cone with volume of 50 cubic inches.

65. Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

5.8 Modeling Using Variation

Learning Objectives

In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

A pre-owned car dealer has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. As she considers the offer, she takes into account the typical price of the dealer's cars, the overall market, and how many she can reasonably expect to sell. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

Solving Direct Variation Problems

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, e, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See Table 1.

s, sales price	$e = 0.16s$	Interpretation
$4,600	$e = 0.16\,(4{,}600) = 736$	A sale of a $4,600 vehicle results in $736 earnings.
$9,200	$e = 0.16\,(9{,}200) = 1{,}472$	A sale of a $9,200 vehicle results in $1472 earnings.
$18,400	$e = 0.16\,(18{,}400) = 2{,}944$	A sale of a $18,400 vehicle results in $2944 earnings.

Table 1

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from $4,600 to $9,200, and we double the earnings from $736 to $1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the

car. The formula $y = kx^n$ is used for direct variation. The value k is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$. We saw functions like this one when we discussed power functions.

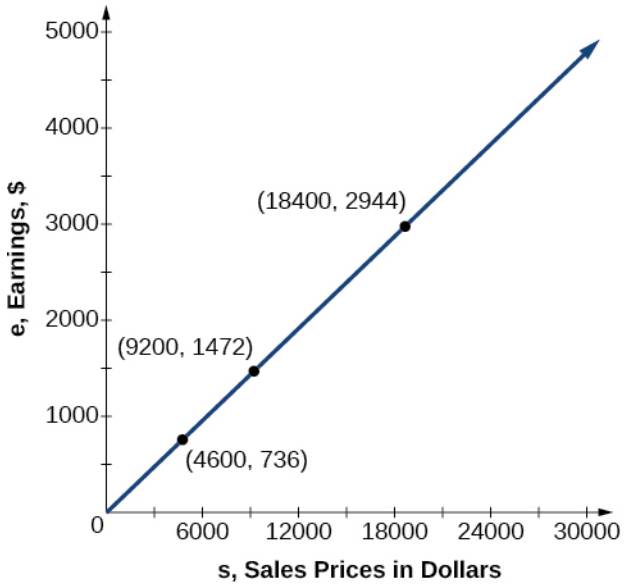


Figure 1

Direct Variation

If x and y are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is **direct variation** and y **varies directly** with, or is proportional to, the nth power of x. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the **constant of variation**, which helps to define the relationship between the variables.

HOW TO

Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

EXAMPLE 1

Solving a Direct Variation Problem

The quantity y varies directly with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution

The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing y by the cube of x.

$$\begin{aligned} k &= \frac{y}{x^3} \\ &= \frac{25}{2^3} \\ &= \frac{25}{8} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{25}{8}(6)^3 \\ &= 675 \end{aligned}$$

Analysis

The graph of this equation is a simple cubic, as shown in Figure 2.

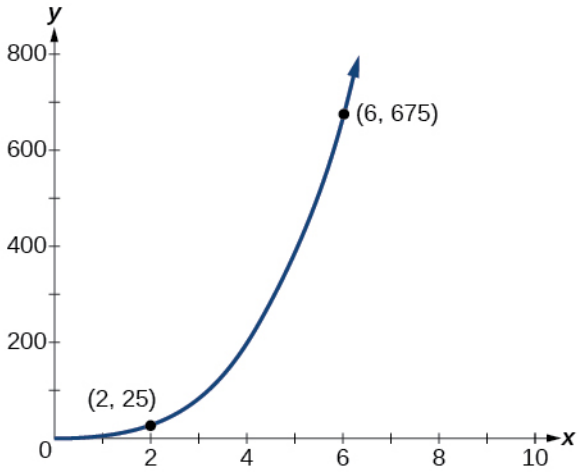


Figure 2

Q&A **Do the graphs of all direct variation equations look like Example 1?**

No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through $(0,0)$.

TRY IT #1 The quantity y varies directly with the square of x. If $y = 24$ when $x = 3$, find y when x is 4.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. The formula $T = \frac{14{,}000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create Table 2, we observe that, as the depth increases, the water temperature decreases.

d, depth	$T = \frac{14{,}000}{d}$	Interpretation
500 ft	$\frac{14{,}000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.
1000 ft	$\frac{14{,}000}{1000} = 14$	At a depth of 1,000 ft, the water temperature is 14° F.
2000 ft	$\frac{14{,}000}{2000} = 7$	At a depth of 2,000 ft, the water temperature is 7° F.

Table 2

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, Figure 3 depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14{,}000$.

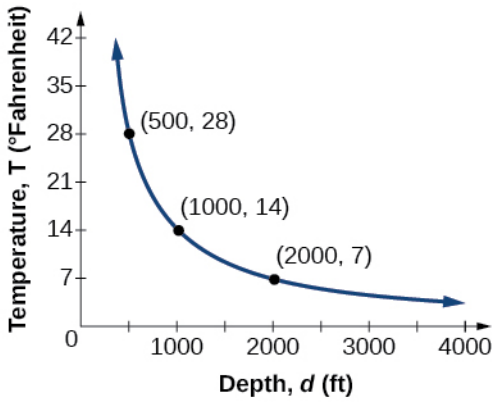


Figure 3

Inverse Variation

If x and y are related by an equation of the form

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y **varies inversely** with the nth power of x. In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

EXAMPLE 2

Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

Solution

Recall that multiplying speed by time gives distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which the tourist drives, then $vt = \text{distance}$. Because the distance is fixed at 100 miles, $vt = 100$ so $t = 100/v$. Because time is a function of velocity, we can write $t(v)$.

$$\begin{aligned} t(v) &= \frac{100}{v} \\ &= 100v^{-1} \end{aligned}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction. We say that time varies inversely with velocity.

HOW TO

Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to multiply y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

EXAMPLE 3

Solving an Inverse Variation Problem

A quantity y varies inversely with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution

The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying y by the cube of x.

$$\begin{aligned} k &= x^3 y \\ &= 2^3 \cdot 25 \\ &= 200 \end{aligned}$$

Now we use the constant to write an equation that represents this relationship.

$$\begin{aligned} y &= \frac{k}{x^3}, \quad k = 200 \\ y &= \frac{200}{x^3} \end{aligned}$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{200}{6^3} \\ &= \frac{25}{27} \end{aligned}$$

Analysis

The graph of this equation is a rational function, as shown in Figure 4.

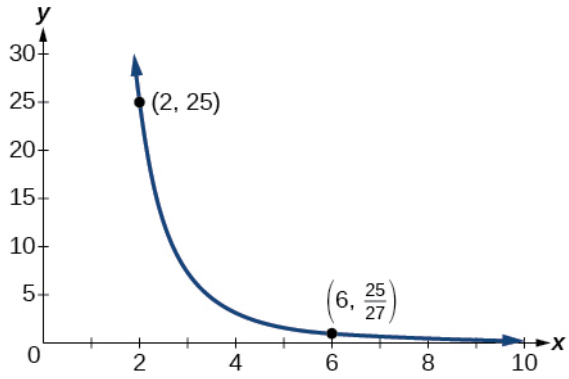


Figure 4

TRY IT #2 A quantity y varies inversely with the square of x. If $y = 8$ when $x = 3$, find y when x is 4.

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c, cost, varies jointly with the number of students, n, and the distance, d.

Joint Variation

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if x varies directly with both y and z, we have $x = kyz$. If x varies directly with y and inversely with z, we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

EXAMPLE 4

Solving Problems Involving Joint Variation

A quantity x varies directly with the square of y and inversely with the cube root of z. If $x = 6$ when $y = 2$ and $z = 8$, find x when $y = 1$ and $z = 27$.

Solution

Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant k.

$$\begin{aligned} 6 &= \frac{k2^2}{\sqrt[3]{8}} \\ 6 &= \frac{4k}{2} \\ 3 &= k \end{aligned}$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find x when $y = 1$ and $z = 27$, we will substitute values for y and z into our equation.

$$\begin{aligned} x &= \frac{3(1)^2}{\sqrt[3]{27}} \\ &= 1 \end{aligned}$$

TRY IT #3 A quantity x varies directly with the square of y and inversely with z. If $x = 40$ when $y = 4$ and $z = 2$, find x when $y = 10$ and $z = 25$.

MEDIA

Access these online resources for additional instruction and practice with direct and inverse variation.

Direct Variation (https://openstax.org/l/directvariation)
Inverse Variation (https://openstax.org/l/inversevariatio)
Direct and Inverse Variation (https://openstax.org/l/directinverse)

5.8 SECTION EXERCISES

Verbal

1. What is true of the appearance of graphs that reflect a direct variation between two variables?

2. If two variables vary inversely, what will an equation representing their relationship look like?

3. Is there a limit to the number of variables that can vary jointly? Explain.

Algebraic

For the following exercises, write an equation describing the relationship of the given variables.

4. y varies directly as x and when $x = 6, y = 12$.

5. y varies directly as the square of x and when $x = 4, \quad y = 80$.

6. y varies directly as the square root of x and when $x = 36, \quad y = 24$.

7. y varies directly as the cube of x and when $x = 36, \quad y = 24$.

8. y varies directly as the cube root of x and when $x = 27, \quad y = 15$.

9. y varies directly as the fourth power of x and when $x = 1, \quad y = 6$.

10. y varies inversely as x and when $x = 4, \quad y = 2$.

11. y varies inversely as the square of x and when $x = 3, \quad y = 2$.

12. y varies inversely as the cube of x and when $x = 2, \quad y = 5$.

13. y varies inversely as the fourth power of x and when $x = 3, \quad y = 1$.

14. y varies inversely as the square root of x and when $x = 25, \quad y = 3$.

15. y varies inversely as the cube root of x and when $x = 64, \quad y = 5$.

16. y varies jointly with x and z and when $x = 2$ and $z = 3, \quad y = 36$.

17. y varies jointly as x, z, and w and when $x = 1, \quad z = 2, \quad w = 5$, then $y = 100$.

18. y varies jointly as the square of x and the square of z and when $x = 3$ and $z = 4$, then $y = 72$.

19. y varies jointly as x and the square root of z and when $x = 2$ and $z = 25$, then $y = 100$.

20. y varies jointly as the square of x the cube of z and the square root of W. When $x = 1, z = 2$, and $w = 36$, then $y = 48$.

21. y varies jointly as x and z and inversely as w. When $x = 3, \quad z = 5$, and $w = 6$, then $y = 10$.

22. y varies jointly as the square of x and the square root of z and inversely as the cube of w. When $x = 3, z = 4$, and $w = 3$, then $y = 6$.

23. y varies jointly as x and z and inversely as the square root of w and the square of t. When $x = 3, z = 1, w = 25$, and $t = 2$, then $y = 6$.

Numeric

For the following exercises, use the given information to find the unknown value.

24. y varies directly as x. When $x = 3$, then $y = 12$. Find y wneh $x = 20$.

25. y varies directly as the square of x. When $x = 2$, then $y = 16$. Find y when $x = 8$.

26. y varies directly as the cube of x. When $x = 3$, then $y = 5$. Find y when $x = 4$.

27. y varies directly as the square root of x. When $x = 16$, then $y = 4$. Find y when $x = 36$.

28. y varies directly as the cube root of x. When $x = 125$, then $y = 15$. Find y when $x = 1{,}000$.

29. y varies inversely with x. When $x = 3$, then $y = 2$. Find y when $x = 1$.

30. y varies inversely with the square of x. When $x = 4$, then $y = 3$. Find y when $x = 2$.

31. y varies inversely with the cube of x. When $x = 3$, then $y = 1$. Find y when $x = 1$.

32. y varies inversely with the square root of x. When $x = 64$, then $y = 12$. Find y when $x = 36$.

33. y varies inversely with the cube root of x. When $x = 27$, then $y = 5$. Find y when $x = 125$.

34. y varies jointly as x and z. When $x = 4$ and $z = 2$, then $y = 16$. Find y when $x = 3$ and $z = 3$.

35. y varies jointly as x, z, and w. When $x = 2$, $z = 1$, and $w = 12$, then $y = 72$. Find y when $x = 1$, $z = 2$, and $w = 3$.

36. y varies jointly as x and the square of z. When $x = 2$ and $z = 4$, then $y = 144$. Find y when $x = 4$ and $z = 5$.

37. y varies jointly as the square of x and the square root of z. When $x = 2$ and $z = 9$, then $y = 24$. Find y when $x = 3$ and $z = 25$.

38. y varies jointly as x and z and inversely as w. When $x = 5$, $z = 2$, and $w = 20$, then $y = 4$. Find y when $x = 3$ and $z = 8$, and $w = 48$.

39. y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x = 2$, $z = 2$, and $w = 64$, then $y = 12$. Find y when $x = 1$, $z = 3$, and $w = 4$.

40. y varies jointly as the square of x and of z and inversely as the square root of w and of t. When $x = 2$, $z = 3$, $w = 16$, and $t = 3$, then $y = 1$. Find y when $x = 3$, $z = 2$, $w = 36$, and $t = 5$.

Technology

For the following exercises, use a calculator to graph the equation implied by the given variation.

41. y varies directly with the square of x and when $x = 2, y = 3$.

42. y varies directly as the cube of x and when $x = 2, y = 4$.

43. y varies directly as the square root of x and when $x = 36, y = 2$.

44. y varies inversely with x and when $x = 6, y = 2$.

45. y varies inversely as the square of x and when $x = 1, y = 4$.

Extensions

For the following exercises, use Kepler's Law, which states that the square of the time, T, required for a planet to orbit the Sun varies directly with the cube of the mean distance, a, that the planet is from the Sun.

46. Using Earth's time of 1 year and mean distance of 93 million miles, find the equation relating T and a.

47. Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

48. Using Earth's distance of 150 million kilometers, find the equation relating T and a.

49. Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

50. Using Earth's distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

Real-World Applications

For the following exercises, use the given information to answer the questions.

51. The distance s that an object falls varies directly with the square of the time, t, of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

52. The velocity v of a falling object varies directly to the time, t, of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

53. The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

54. The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

55. The weight of an object above the surface of Earth varies inversely with the square of the distance from the center of Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

56. The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is 0.08 foot-candles at a distance of 3 meters. Find the intensity level at 8 meters.

57. The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

58. The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

59. The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

60. The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 40 kilograms with a velocity of 15 meters per second has a kinetic energy of 1000 joules, find the kinetic energy if the velocity is increased to 20 meters per second.

Chapter Review

Key Terms

arrow notation a way to represent symbolically the local and end behavior of a function by using arrows to indicate that an input or output approaches a value

axis of symmetry a vertical line drawn through the vertex of a parabola, that opens up or down, around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

coefficient a nonzero real number multiplied by a variable raised to an exponent

constant of variation the non-zero value k that helps define the relationship between variables in direct or inverse variation

continuous function a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree the highest power of the variable that occurs in a polynomial

Descartes' Rule of Signs a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

direct variation the relationship between two variables that are a constant multiple of each other; as one quantity increases, so does the other

Division Algorithm given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

end behavior the behavior of the graph of a function as the input decreases without bound and increases without bound

Factor Theorem k is a zero of polynomial function $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra a polynomial function with degree greater than 0 has at least one complex zero

general form of a quadratic function the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

global maximum highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

global minimum lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

horizontal asymptote a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound.

Intermediate Value Theorem for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x- axis

inverse variation the relationship between two variables in which the product of the variables is a constant

inversely proportional a relationship where one quantity is a constant divided by the other quantity; as one quantity increases, the other decreases

invertible function any function that has an inverse function

joint variation a relationship where a variable varies directly or inversely with multiple variables

leading coefficient the coefficient of the leading term

leading term the term containing the highest power of the variable

Linear Factorization Theorem allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number

multiplicity the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x - h)^p$, $x = h$ is a zero of multiplicity p.

polynomial function a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

rational function a function that can be written as the ratio of two polynomials

Rational Zero Theorem the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem if a polynomial $f(x)$ is divided by $x - k$, then the remainder is equal to the value $f(k)$

removable discontinuity a single point at which a function is undefined that, if filled in, would make the function continuous; it appears as a hole on the graph of a function

roots in a given function, the values of x at which $y = 0$, also called zeros

smooth curve a graph with no sharp corners
standard form of a quadratic function the function that describes a parabola, written in the form $f(x) = a(x-h)^2 + k$, where $(h, \ k)$ is the vertex
synthetic division a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$
term of a polynomial function any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$
turning point the location at which the graph of a function changes direction
varies directly a relationship where one quantity is a constant multiplied by the other quantity
varies inversely a relationship where one quantity is a constant divided by the other quantity
vertex the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function
vertex form of a quadratic function another name for the standard form of a quadratic function
vertical asymptote a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a
zeros in a given function, the values of x at which $y = 0$, also called roots

Key Equations

general form of a quadratic function	$f(x) = ax^2 + bx + c$
standard form of a quadratic function	$f(x) = a(x-h)^2 + k$
general form of a polynomial function	$f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$
Division Algorithm	$f(x) = d(x)q(x) + r(x) \ \text{ where } q(x) \neq 0$
Rational Function	$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + ... + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + ... + b_1 x + b_0}, \ Q(x) \neq 0$
Direct variation	$y = kx^n, \ \ k$ is a nonzero constant.
Inverse variation	$y = \frac{k}{x^n}, \ \ k$ is a nonzero constant.

Key Concepts

5.1 Quadratic Functions

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x- intercepts, are the points at which the parabola crosses the x- axis. The y- intercept is the point at which the parabola crosses the y- axis. See Example 1, Example 7, and Example 8.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph. See Example 2.
- The vertex can be found from an equation representing a quadratic function. See Example 3.
- The domain of a quadratic function is all real numbers. The range varies with the function. See Example 4.
- A quadratic function's minimum or maximum value is given by the y- value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue. See Example 5 and Example 6.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems. See Example 9.

5.2 Power Functions and Polynomial Functions

- A power function is a variable base raised to a number power. See Example 1.
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd. See Example 2 and Example 3.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power. See Example 4.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient. See Example 5.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function. See Example 6 and Example 7.
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points. See Example 8, Example 9, Example 10, Example 11, and Example 12.

5.3 Graphs of Polynomial Functions

- Polynomial functions of degree 2 or more are smooth, continuous functions. See Example 1.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero. See Example 2, Example 3, and Example 4.
- Another way to find the x- intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x- axis. See Example 5.
- The multiplicity of a zero determines how the graph behaves at the x- intercepts. See Example 6.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points. See Example 7.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points. See Example 8 and Example 10.
- Graphing a polynomial function helps to estimate local and global extremas. See Example 11.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$. See Example 9.

5.4 Dividing Polynomials

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree. See Example 1 and Example 2.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x-k$. See Example 3, Example 4, and Example 5.
- Polynomial division can be used to solve application problems, including area and volume. See Example 6.

5.5 Zeros of Polynomial Functions

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x-k$. This is known as the Remainder Theorem. See Example 1.
- According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$. See Example 2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 3 and Example 4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 5.
- According to the Fundamental Theorem, every polynomial function has at least one complex zero. See Example 6.
- Every polynomial function with degree greater than 0 has at least one complex zero.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x-c)$, where c is a complex number. See Example 7.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less

than the number of sign changes by an even integer. See Example 8.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 9.

5.6 Rational Functions

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example 1.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example 2.
- Application problems involving rates and concentrations often involve rational functions. See Example 3.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example 4.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example 5.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example 6.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example 7, Example 8, Example 9, and Example 10.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example 11.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i = \text{any } v_j$, then the function can be written in the form

$$f(x) = a\frac{(x-x_1)^{p1}(x-x_2)^{p2}\cdots(x-x_n)^{pn}}{(x-v_1)^{q1}(x-v_2)^{q2}\cdots(x-v_m)^{qn}}$$

See Example 12.

5.7 Inverses and Radical Functions

- The inverse of a quadratic function is a square root function.
- If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. See Example 1.
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See Example 2.
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one. See Example 3 and Example 4.
- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See Example 5 and Example 7.
- Inverse and radical and functions can be used to solve application problems. See Example 6 and Example 8.

5.8 Modeling Using Variation

- A relationship where one quantity is a constant multiplied by another quantity is called direct variation. See Example 1.
- Two variables that are directly proportional to one another will have a constant ratio.
- A relationship where one quantity is a constant divided by another quantity is called inverse variation. See Example 2.
- Two variables that are inversely proportional to one another will have a constant multiple. See Example 3.
- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation. See Example 4.

Exercises

Review Exercises

Quadratic Functions

For the following exercises, write the quadratic function in standard form. Then give the vertex and axes intercepts. Finally, graph the function.

1. $f(x) = x^2 - 4x - 5$

2. $f(x) = -2x^2 - 4x$

For the following exercises, find the equation of the quadratic function using the given information.

3. The vertex is $(-2, 3)$ and a point on the graph is $(3, 6)$.

4. The vertex is $(-3, 6.5)$ and a point on the graph is $(2, 6)$.

For the following exercises, complete the task.

5. A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is 600 meters, find the dimensions of the plot to have maximum area.

6. An object projected from the ground at a 45 degree angle with initial velocity of 120 feet per second has height, h, in terms of horizontal distance traveled, x, given by $h(x) = \frac{-32}{(120)^2}x^2 + x$. Find the maximum height the object attains.

Power Functions and Polynomial Functions

For the following exercises, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.

7. $f(x) = 4x^5 - 3x^3 + 2x - 1$

8. $f(x) = 5^{x+1} - x^2$

9. $f(x) = x^2\left(3 - 6x + x^2\right)$

For the following exercises, determine end behavior of the polynomial function.

10. $f(x) = 2x^4 + 3x^3 - 5x^2 + 7$

11. $f(x) = 4x^3 - 6x^2 + 2$

12. $f(x) = 2x^2(1 + 3x - x^2)$

Graphs of Polynomial Functions

For the following exercises, find all zeros of the polynomial function, noting multiplicities.

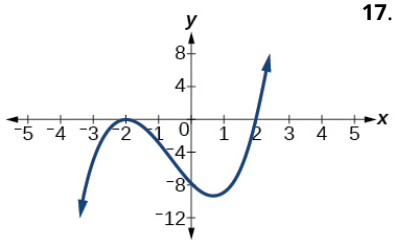
13. $f(x) = (x + 3)^2(2x - 1)(x + 1)^3$

14. $f(x) = x^5 + 4x^4 + 4x^3$

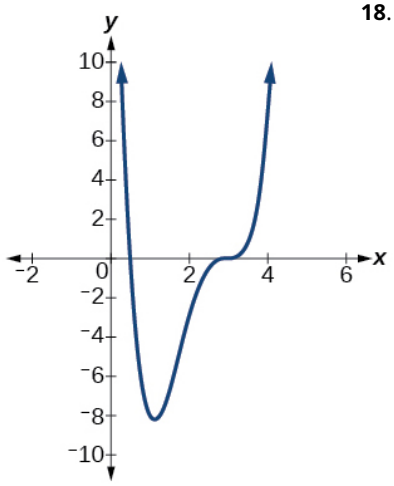
15. $f(x) = x^3 - 4x^2 + x - 4$

For the following exercises, based on the given graph, determine the zeros of the function and note multiplicity.

16.



17.



18. Use the Intermediate Value Theorem to show that at least one zero lies between 2 and 3 for the function $f(x) = x^3 - 5x + 1$

Dividing Polynomials

For the following exercises, use long division to find the quotient and remainder.

19. $\frac{x^3-2x^2+4x+4}{x-2}$

20. $\frac{3x^4-4x^2+4x+8}{x+1}$

For the following exercises, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.

21. $\frac{x^3-2x^2+5x-1}{x+3}$

22. $\frac{x^3+4x+10}{x-3}$

23. $\frac{2x^3+6x^2-11x-12}{x+4}$

24. $\frac{3x^4+3x^3+2x+2}{x+1}$

Zeros of Polynomial Functions

For the following exercises, use the Rational Zero Theorem to help you solve the polynomial equation.

25. $2x^3 - 3x^2 - 18x - 8 = 0$ **26.** $3x^3 + 11x^2 + 8x - 4 = 0$ **27.** $2x^4 - 17x^3 + 46x^2 - 43x + 12 = 0$

28. $4x^4 + 8x^3 + 19x^2 + 32x + 12 = 0$

For the following exercises, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.

29. $x^3 - 3x^2 - 2x + 4 = 0$ **30.** $2x^4 - x^3 + 4x^2 - 5x + 1 = 0$

Rational Functions

For the following exercises, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph of the function.

31. $f(x) = \frac{x+2}{x-5}$ **32.** $f(x) = \frac{x^2+1}{x^2-4}$ **33.** $f(x) = \frac{3x^2-27}{x^2+x-2}$

34. $f(x) = \frac{x+2}{x^2-9}$

For the following exercises, find the slant asymptote.

35. $f(x) = \frac{x^2-1}{x+2}$ **36.** $f(x) = \frac{2x^3-x^2+4}{x^2+1}$

Inverses and Radical Functions

For the following exercises, find the inverse of the function with the domain given.

37. $f(x) = (x-2)^2, \quad x \geq 2$ **38.** $f(x) = (x+4)^2 - 3, \quad x \geq -4$ **39.** $f(x) = x^2 + 6x - 2, \quad x \geq -3$

40. $f(x) = 2x^3 - 3$ **41.** $f(x) = \sqrt{4x+5} - 3$ **42.** $f(x) = \frac{x-3}{2x+1}$

Modeling Using Variation

For the following exercises, find the unknown value.

43. y varies directly as the square of x. If when $x = 3, \quad y = 36$, find y if $x = 4$.

44. y varies inversely as the square root of x If when $x = 25, \quad y = 2$, find y if $x = 4$.

45. y varies jointly as the cube of x and as z. If when $x = 1$ and $z = 2$, $y = 6$, find y if $x = 2$ and $z = 3$.

46. y varies jointly as x and the square of z and inversely as the cube of w. If when $x = 3$, $z = 4$, and $w = 2$, $y = 48$, find y if $x = 4$, $z = 5$, and $w = 3$.

For the following exercises, solve the application problem.

47. The weight of an object above the surface of the earth varies inversely with the square of the distance from the center of the earth. If a person weighs 150 pounds when he is on the surface of the earth (3,960 miles from center), find the weight of the person if he is 20 miles above the surface.

48. The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure P. A cylinder contains oxygen at a temperature of 310 degrees K and a pressure of 18 atmospheres in a volume of 120 liters. Find the pressure if the volume is decreased to 100 liters and the temperature is increased to 320 degrees K.

Practice Test

Give the degree and leading coefficient of the following polynomial function.

1. $f(x) = x^3\left(3 - 6x - 2x^2\right)$

Determine the end behavior of the polynomial function.

2. $f(x) = 8x^3 - 3x^2 + 2x - 4$ **3.** $f(x) = -2x^2(4 - 3x - 5x^2)$

Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

4. $f(x) = x^2 + 2x - 8$

Given information about the graph of a quadratic function, find its equation.

5. Vertex $(2, 0)$ and point on graph $(4, 12)$.

Solve the following application problem.

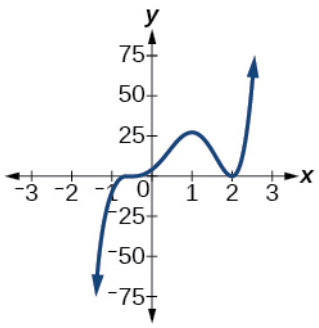
6. A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If 1,200 feet of fencing is available, find the maximum area that can be enclosed.

Find all zeros of the following polynomial functions, noting multiplicities.

7. $f(x) = (x - 3)^3(3x - 1)(x - 1)^2$ **8.** $f(x) = 2x^6 - 12x^5 + 18x^4$

Based on the graph, determine the zeros of the function and multiplicities.

9.



Use long division to find the quotient.

10. $\frac{2x^3+3x-4}{x+2}$

Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.

11. $\frac{x^4+3x^2-4}{x-2}$ **12.** $\frac{2x^3+5x^2-7x-12}{x+3}$

Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.

13. $f(x) = 2x^3 + 5x^2 - 6x - 9$ **14.** $f(x) = 4x^4 + 8x^3 + 21x^2 + 17x + 4$

15. $f(x) = 4x^4 + 16x^3 + 13x^2 - 15x - 18$

16. $f(x) = x^5 + 6x^4 + 13x^3 + 14x^2 + 12x + 8$

Given the following information about a polynomial function, find the function.

17. It has a double zero at $x = 3$ and zeros at $x = 1$ and $x = -2$. Its *y*-intercept is $(0, 12)$.

18. It has a zero of multiplicity 3 at $x = \frac{1}{2}$ and another zero at $x = -3$. It contains the point $(1, 8)$.

Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

19. $8x^3 - 21x^2 + 6 = 0$

For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.

20. $f(x) = \frac{x+4}{x^2-2x-3}$

21. $f(x) = \frac{x^2+2x-3}{x^2-4}$

Find the slant asymptote of the rational function.

22. $f(x) = \frac{x^2+3x-3}{x-1}$

Find the inverse of the function.

23. $f(x) = \sqrt{x-2} + 4$

24. $f(x) = 3x^3 - 4$

25. $f(x) = \frac{2x+3}{3x-1}$

Find the unknown value.

26. y varies inversely as the square of x and when $x = 3$, $y = 2$. Find y if $x = 1$.

27. y varies jointly with x and the cube root of z. If when $x = 2$ and $z = 27$, $y = 12$, find y if $x = 5$ and $z = 8$.

Solve the following application problem.

28. The distance a body falls varies directly as the square of the time it falls. If an object falls 64 feet in 2 seconds, how long will it take to fall 256 feet?