

2 EQUATIONS AND INEQUALITIES

From the air, a landscape of circular crop fields may seem random, but they are laid out and irrigated very precisely. Farmers and irrigation providers combining agricultural science, engineering, and mathematics to achieve the most productive and efficient array. (Credit: Modification of "Aerial Phot of Center Pivot Irrigations Systems (1)" by Soil Science/ flickr)

Chapter Outline

2.1 The Rectangular Coordinate Systems and Graphs
2.2 Linear Equations in One Variable
2.3 Models and Applications
2.4 Complex Numbers
2.5 Quadratic Equations
2.6 Other Types of Equations
2.7 Linear Inequalities and Absolute Value Inequalities

Introduction to Equations and Inequalities

Irrigation is a critical aspect of agriculture, which can expand the yield of farms and enable farming in areas not naturally viable for crops. But the materials, equipment, and the water itself are expensive and complex. To be efficient and productive, farm owners and irrigation specialists must carefully lay out the network of pipes, pumps, and related equipment. The available land can be divided into regular portions (similar to a grid), and the different sizes of irrigation systems and conduits can be installed within the plotted area.

2.1 The Rectangular Coordinate Systems and Graphs

Learning Objectives

In this section, you will:

- Plot ordered pairs in a Cartesian coordinate system.
- Graph equations by plotting points.
- Graph equations with a graphing utility.
- Find x-intercepts and y-intercepts.
- Use the distance formula.
- Use the midpoint formula.

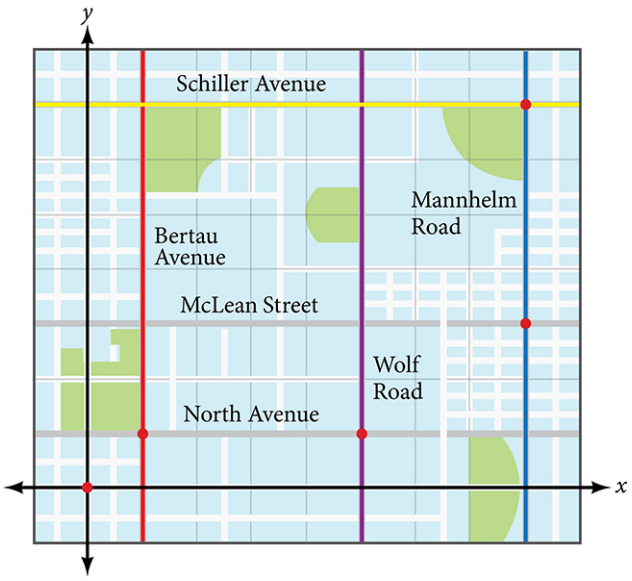


Figure 1

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure 1. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

Plotting Ordered Pairs in the Cartesian Coordinate System

An old story describes how seventeenth-century philosopher/mathematician René Descartes, while sick in bed, invented the system that has become the foundation of algebra. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the **Cartesian coordinate system**, a grid system having perpendicular axes. Descartes named the horizontal axis the ***x*-axis** and the vertical axis the ***y*-axis**.

The Cartesian coordinate system, also called the rectangular coordinate system, is based on a two-dimensional plane consisting of the *x*-axis and the *y*-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a **quadrant**; the quadrants are numbered counterclockwise as shown in Figure 2

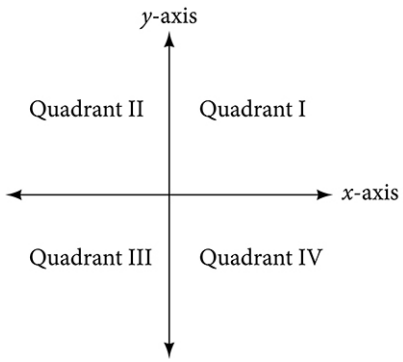


Figure 2

The center of the plane is the point at which the two axes cross. It is known as the **origin**, or point $(0, 0)$. From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the *x*-axis and up the *y*-axis; decreasing, negative numbers to the left on the *x*-axis and down the *y*-axis. The axes extend to positive and negative infinity as shown by the arrowheads in Figure 3.

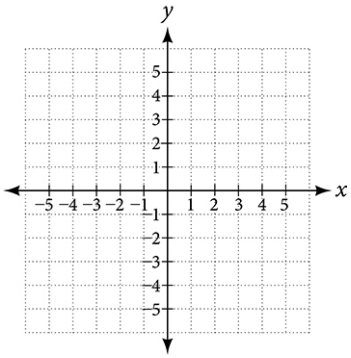


Figure 3

Each point in the plane is identified by its ***x*-coordinate**, or horizontal displacement from the origin, and its ***y*-coordinate**, or vertical displacement from the origin. Together, we write them as an **ordered pair** indicating the combined distance from the origin in the form (x, y). An ordered pair is also known as a coordinate pair because it consists of *x*- and *y*-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See Figure 4.

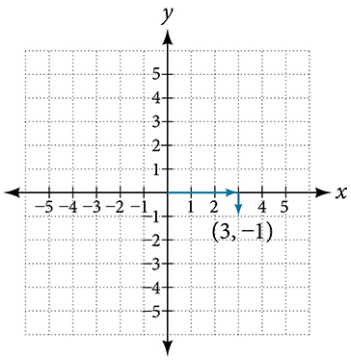


Figure 4

When dividing the axes into equally spaced increments, note that the *x*-axis may be considered separately from the *y*-axis. In other words, while the *x*-axis may be divided and labeled according to consecutive integers, the *y*-axis may be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Cartesian Coordinate System

A two-dimensional plane where the

- *x*-axis is the horizontal axis
- *y*-axis is the vertical axis

A point in the plane is defined as an ordered pair, (x, y), such that *x* is determined by its horizontal distance from the origin and *y* is determined by its vertical distance from the origin.

EXAMPLE 1

Plotting Points in a Rectangular Coordinate System

Plot the points $(-2, 4)$, $(3, 3)$, and $(0, -3)$ in the plane.

Solution

To plot the point $(-2, 4)$, begin at the origin. The *x*-coordinate is –2, so move two units to the left. The *y*-coordinate is 4, so then move four units up in the positive *y* direction.

To plot the point $(3, 3)$, begin again at the origin. The *x*-coordinate is 3, so move three units to the right. The *y*-coordinate is also 3, so move three units up in the positive *y* direction.

To plot the point $(0, -3)$, begin again at the origin. The *x*-coordinate is 0. This tells us not to move in either direction along the *x*-axis. The *y*-coordinate is –3, so move three units down in the negative *y* direction. See the graph in Figure 5.

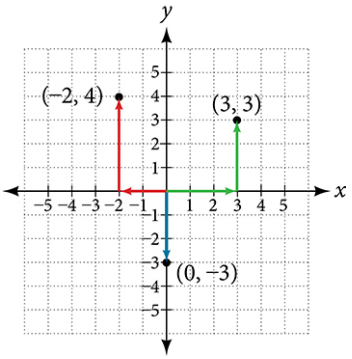


Figure 5

Analysis

Note that when either coordinate is zero, the point must be on an axis. If the *x*-coordinate is zero, the point is on the *y*-axis. If the *y*-coordinate is zero, the point is on the *x*-axis.

Graphing Equations by Plotting Points

We can plot a set of points to represent an equation. When such an equation contains both an *x* variable and a *y* variable, it is called an **equation in two variables**. Its graph is called a **graph in two variables**. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for *x* into the equation and determining the resulting value of *y*. Each pair of *x*- and *y*-values is an ordered pair that can be plotted. Table 1 lists values of *x* from –3 to 3 and the resulting values for *y*.

x	$y = 2x - 1$	(x, y)
-3	$y = 2(-3) - 1 = -7$	$(-3, -7)$
-2	$y = 2(-2) - 1 = -5$	$(-2, -5)$
-1	$y = 2(-1) - 1 = -3$	$(-1, -3)$
0	$y = 2(0) - 1 = -1$	$(0, -1)$
1	$y = 2(1) - 1 = 1$	$(1, 1)$
2	$y = 2(2) - 1 = 3$	$(2, 3)$
3	$y = 2(3) - 1 = 5$	$(3, 5)$

Table 1

We can plot the points in the table. The points for this particular equation form a line, so we can connect them. See Figure 6**.** This is not true for all equations.

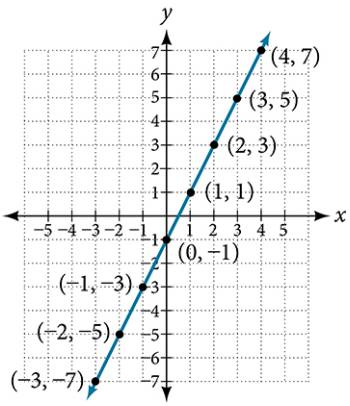


Figure 6

Note that the *x*-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of *x* to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

HOW TO

Given an equation, graph by plotting points.

1. Make a table with one column labeled *x*, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter *x*-values down the first column using positive and negative values. Selecting the *x*-values in numerical order will make the graphing simpler.
3. Select *x*-values that will yield *y*-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

EXAMPLE 2

Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y = -x + 2$ by plotting points.

Solution

First, we construct a table similar to Table 2. Choose *x* values and calculate *y*.

x	$y = -x + 2$	(x, y)
-5	$y = -(-5) + 2 = 7$	$(-5, 7)$
-3	$y = -(-3) + 2 = 5$	$(-3, 5)$
-1	$y = -(-1) + 2 = 3$	$(-1, 3)$
0	$y = -(0) + 2 = 2$	$(0, 2)$
1	$y = -(1) + 2 = 1$	$(1, 1)$

Table 2

3	$y = -(3) + 2 = -1$	$(3, -1)$
5	$y = -(5) + 2 = -3$	$(5, -3)$

Table 2

Now, plot the points. Connect them if they form a line. See Figure 7

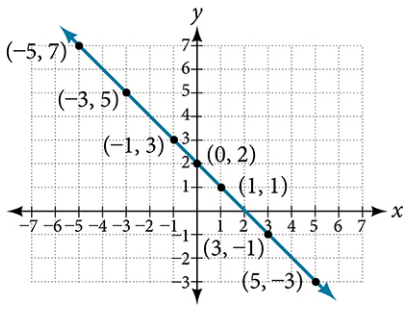


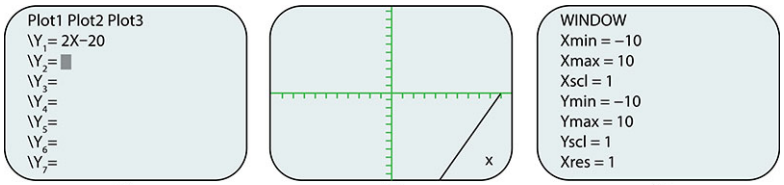
Figure 7

TRY IT #1 Construct a table and graph the equation by plotting points: $y = \frac{1}{2}x + 2$.

Graphing Equations with a Graphing Utility

Most graphing calculators require similar techniques to graph an equation. The equations sometimes have to be manipulated so they are written in the style $y =$_____. The TI-84 Plus, and many other calculator makes and models, have a mode function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

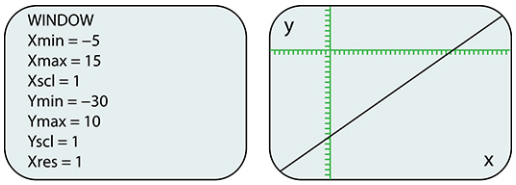
For example, the equation $y = 2x - 20$ has been entered in the TI-84 Plus shown in Figure 8**a.** In Figure 8**b,** the resulting graph is shown. Notice that we cannot see on the screen where the graph crosses the axes. The standard window screen on the TI-84 Plus shows $-10 \leq x \leq 10$, and $-10 \leq y \leq 10$. See Figure 8**c**.



(a) (b) (c)

Figure 8 a. Enter the equation. b. This is the graph in the original window. c. These are the original settings.

By changing the window to show more of the positive *x*-axis and more of the negative *y*-axis, we have a much better view of the graph and the *x*- and *y*-intercepts. See Figure 9**a** and Figure 9**b.**



(a) (b)

Figure 9 a. This screen shows the new window settings. b. We can clearly view the intercepts in the new window.

EXAMPLE 3

Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph the equation: $y = -\frac{2}{3}x + \frac{4}{3}$.

Solution

Enter the equation in the *y*= function of the calculator. Set the window settings so that both the *x*- and *y*- intercepts are showing in the window. See Figure 10.

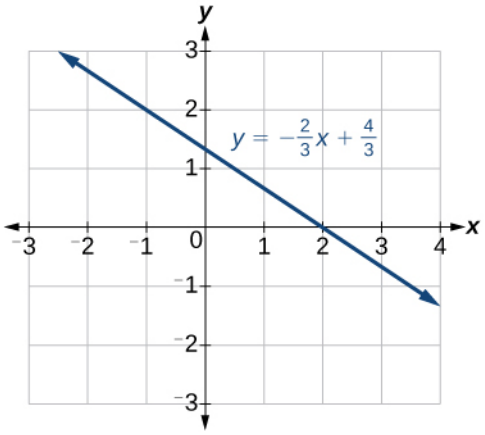


Figure 10

Finding *x*-intercepts and *y*-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The ***x*-intercept** is the point at which the graph crosses the *x*-axis. At this point, the *y*-coordinate is zero. The ***y*-intercept** is the point at which the graph crosses the *y*-axis. At this point, the *x*-coordinate is zero.

To determine the *x*-intercept, we set *y* equal to zero and solve for *x*. Similarly, to determine the *y*-intercept, we set *x* equal to zero and solve for *y*. For example, lets find the intercepts of the equation $y = 3x - 1$.

To find the *x*-intercept, set $y = 0$.

$$\begin{aligned} y &= 3x - 1 \\ 0 &= 3x - 1 \\ 1 &= 3x \\ \tfrac{1}{3} &= x \\ \left(\tfrac{1}{3}, 0\right) &\qquad x\text{−intercept} \end{aligned}$$

To find the *y*-intercept, set $x = 0$.

$$\begin{aligned} y &= 3x - 1 \\ y &= 3(0) - 1 \\ y &= -1 \\ (0, -1) &\qquad y\text{−intercept} \end{aligned}$$

We can confirm that our results make sense by observing a graph of the equation as in Figure 11. Notice that the graph crosses the axes where we predicted it would.

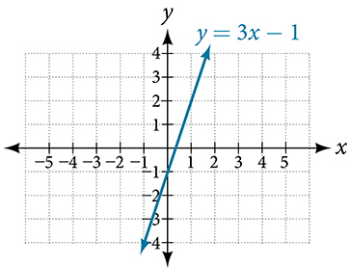


Figure 11

Given an equation, find the intercepts.

- Find the *x*-intercept by setting $y = 0$ and solving for x.
- Find the *y*-intercept by setting $x = 0$ and solving for y.

EXAMPLE 4

Finding the Intercepts of the Given Equation

Find the intercepts of the equation $y = -3x - 4$. Then sketch the graph using only the intercepts.

Solution

Set $y = 0$ to find the *x*-intercept.

$$\begin{aligned} y &= -3x - 4 \\ 0 &= -3x - 4 \\ 4 &= -3x \\ -\tfrac{4}{3} &= x \\ \left(-\tfrac{4}{3}, 0\right) &\qquad x\text{−intercept} \end{aligned}$$

Set $x = 0$ to find the *y*-intercept.

$$\begin{aligned} y &= -3x - 4 \\ y &= -3(0) - 4 \\ y &= -4 \\ (0, -4) &\qquad y\text{−intercept} \end{aligned}$$

Plot both points, and draw a line passing through them as in Figure 12.

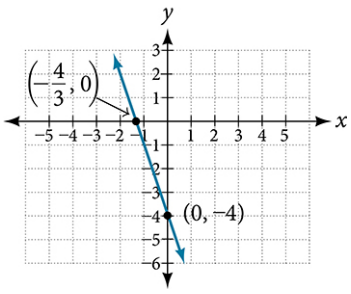


Figure 12

TRY IT #2 Find the intercepts of the equation and sketch the graph: $y = -\frac{3}{4}x + 3$.

Using the Distance Formula

Derived from the Pythagorean Theorem, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2 + b^2 = c^2$, is based on a right triangle where *a* and *b* are the lengths of the legs

adjacent to the right angle, and *c* is the length of the hypotenuse. See Figure 13.

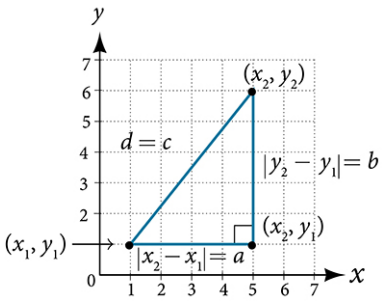


Figure 13

The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side *d* is the same as that of sides *a* and *b* to side *c*. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3$.) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length *c*, take the square root of both sides of the Pythagorean Theorem.

$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2}$$

It follows that the distance formula is given as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

The Distance Formula

Given endpoints (x_1, y_1) and (x_2, y_2), the distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 5

Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

Solution

Let us first look at the graph of the two points. Connect the points to form a right triangle as in Figure 14.

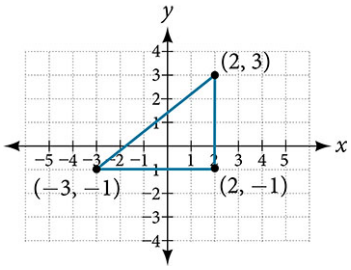


Figure 14

Then, calculate the length of *d* using the distance formula.

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d &= \sqrt{(2 - (-3))^2 + (3 - (-1))^2} \\
&= \sqrt{(5)^2 + (4)^2} \\
&= \sqrt{25 + 16} \\
&= \sqrt{41}
\end{aligned}
$$

TRY IT #3 Find the distance between two points: $(1, 4)$ and $(11, 9)$.

EXAMPLE 6

Finding the Distance between Two Locations

Let's return to the situation introduced at the beginning of this section.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure 1. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.

Solution

The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1, 1)$. The next stop is 5 blocks to the east, so it is at $(5, 1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8, 3)$. Lastly, she traveled 4 blocks north to $(8, 7)$. We can label these points on the grid as in Figure 15.

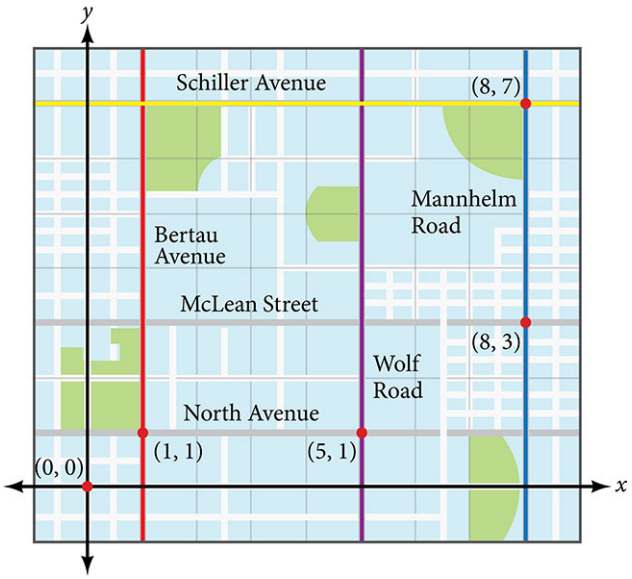


Figure 15

Next, we can calculate the distance. Note that each grid unit represents 1,000 feet.

- From her starting location to her first stop at $(1, 1)$, Tracie might have driven north 1,000 feet and then east 1,000 feet, or vice versa. Either way, she drove 2,000 feet to her first stop.
- Her second stop is at $(5, 1)$. So from $(1, 1)$ to $(5, 1)$, Tracie drove east 4,000 feet.
- Her third stop is at $(8, 3)$. There are a number of routes from $(5, 1)$ to $(8, 3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east 3,000 feet and then north 2,000 feet for a total of 5,000 feet.
- Tracie's final stop is at $(8, 7)$. This is a straight drive north from $(8, 3)$ for a total of 4,000 feet.

Next, we will add the distances listed in Table 3.

From/To	Number of Feet Driven
$(0, 0)$ to $(1, 1)$	2,000
$(1, 1)$ to $(5, 1)$	4,000
$(5, 1)$ to $(8, 3)$	5,000
$(8, 3)$ to $(8, 7)$	4,000
Total	15,000

Table 3

The total distance Tracie drove is 15,000 feet, or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0, 0)$ and $(8, 7)$.

$$\begin{aligned} d &= \sqrt{(8-0)^2 + (7-0)^2} \\ &= \sqrt{64 + 49} \\ &= \sqrt{113} \\ &\approx 10.63 \text{ units} \end{aligned}$$

At 1,000 feet per grid unit, the distance between Elmhurst, IL, to Franklin Park is 10,630.14 feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8, 7)$. Perhaps you have heard the saying "as the crow flies," which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

Using the Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the **midpoint formula**. Given the endpoints of a line segment, (x_1, y_1) and (x_2, y_2), the midpoint formula states how to find the coordinates of the midpoint M.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

A graphical view of a midpoint is shown in Figure 16. Notice that the line segments on either side of the midpoint are congruent.

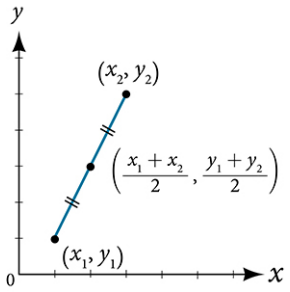


Figure 16

EXAMPLE 7

Finding the Midpoint of the Line Segment

Find the midpoint of the line segment with the endpoints $(7, -2)$ and $(9, 5)$.

Solution

Use the formula to find the midpoint of the line segment.

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{7+9}{2}, \frac{-2+5}{2}\right)$$
$$= \left(8, \frac{3}{2}\right)$$

TRY IT #4 Find the midpoint of the line segment with endpoints $(-2, -1)$ and $(-8, 6)$.

EXAMPLE 8

Finding the Center of a Circle

The diameter of a circle has endpoints $(-1, -4)$ and $(5, -4)$. Find the center of the circle.

Solution

The center of a circle is the center, or midpoint, of its diameter. Thus, the midpoint formula will yield the center point.

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$\left(\frac{-1+5}{2}, \frac{-4-4}{2}\right) = \left(\frac{4}{2}, -\frac{8}{2}\right) = (2, -4)$$

MEDIA

Access these online resources for additional instruction and practice with the Cartesian coordinate system.

Plotting points on the coordinate plane (https://openstax.org/l/coordplotpnts)
Find x and y intercepts based on the graph of a line (https://openstax.org/l/xyintsgraph)

2.1 SECTION EXERCISES

Verbal

1. Is it possible for a point plotted in the Cartesian coordinate system to not lie in one of the four quadrants? Explain.
2. Describe the process for finding the *x*-intercept and the *y*-intercept of a graph algebraically.
3. Describe in your own words what the *y*-intercept of a graph is.
4. When using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, explain the correct order of operations that are to be performed to obtain the correct answer.

Algebraic

For each of the following exercises, find the x-intercept and the y-intercept without graphing. Write the coordinates of each intercept.

5. $y = -3x + 6$

6. $4y = 2x - 1$

7. $3x - 2y = 6$

8. $4x - 3 = 2y$

9. $3x + 8y = 9$

10. $2x - \frac{2}{3} = \frac{3}{4}y + 3$

For each of the following exercises, solve the equation for y in terms of x.

11. $4x + 2y = 8$

12. $3x - 2y = 6$

13. $2x = 5 - 3y$

14. $x - 2y = 7$

15. $5y + 4 = 10x$

16. $5x + 2y = 0$

For each of the following exercises, find the distance between the two points. Simplify your answers, and write the exact answer in simplest radical form for irrational answers.

17. $(-4, 1)$ and $(3, -4)$

18. $(2, -5)$ and $(7, 4)$

19. $(5, 0)$ and $(5, 6)$

20. $(-4, 3)$ and $(10, 3)$

21. Find the distance between the two points given using your calculator, and round your answer to the nearest hundredth.
$(19, 12)$ and $(41, 71)$

For each of the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

22. $(-5, -6)$ and $(4, 2)$

23. $(-1, 1)$ and $(7, -4)$

24. $(-5, -3)$ and $(-2, -8)$

25. $(0, 7)$ and $(4, -9)$

26. $(-43, 17)$ and $(23, -34)$

Graphical

For each of the following exercises, identify the information requested.

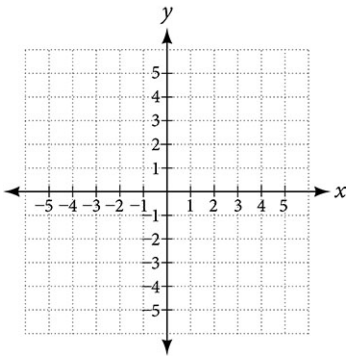
27. What are the coordinates of the origin?

28. If a point is located on the *y*-axis, what is the *x*-coordinate?

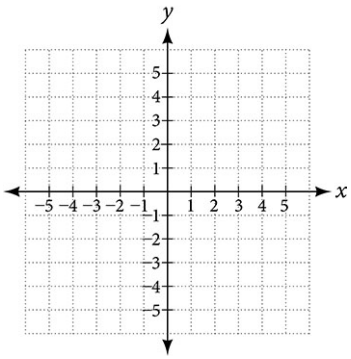
29. If a point is located on the *x*-axis, what is the *y*-coordinate?

For each of the following exercises, plot the three points on the given coordinate plane. State whether the three points you plotted appear to be collinear (on the same line).

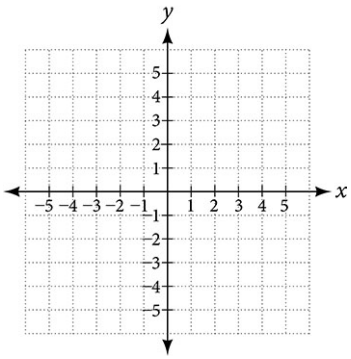
30. $(4, 1)\,(-2, -3)\,(5, 0)$



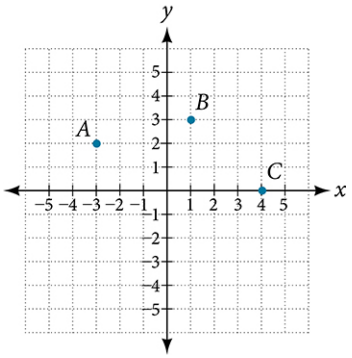
31. $(-1, 2)(0, 4)(2, 1)$



32. $(-3, 0)\,(-3, 4)\,(-3, -3)$



33. Name the coordinates of the points graphed.



34. Name the quadrant in which the following points would be located. If the point is on an axis, name the axis.

ⓐ $(-3,-4)$ ⓑ $(-5,0)$
ⓒ $(1,-4)$ ⓓ $(-2,7)$
ⓔ $(0,-3)$

For each of the following exercises, construct a table and graph the equation by plotting at least three points.

35. $y=\frac{1}{3}x+2$

36. $y=-3x+1$

37. $2y=x+3$

Numeric

For each of the following exercises, find and plot the x- and y-intercepts, and graph the straight line based on those two points.

38. $4x-3y=12$

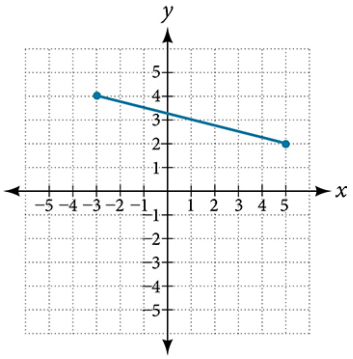
39. $x-2y=8$

40. $y-5=5x$

41. $3y=-2x+6$

42. $y=\frac{x-3}{2}$

For each of the following exercises, use the graph in the figure below.



43. Find the distance between the two endpoints using the distance formula. Round to three decimal places.

44. Find the coordinates of the midpoint of the line segment connecting the two points.

45. Find the distance that $(-3,4)$ is from the origin.

46. Find the distance that $(5,2)$ is from the origin. Round to three decimal places.

47. Which point is closer to the origin?

Technology

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

After graphing it, use the 2nd CALC button and 1:value button, hit enter. At the lower part of the screen you will see "x=" and a blinking cursor. You may enter any number for x and it will display the y value for any x value you input. Use this and plug in x = 0, thus finding the y-intercept, for each of the following graphs.

48. $Y_1 = -2x + 5$

49. $Y_1 = \frac{3x-8}{4}$

50. $Y_1 = \frac{x+5}{2}$

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

*After graphing it, use the 2nd **CALC** button and 2:zero button, hit **ENTER**. At the lower part of the screen you will see "left bound?" and a blinking cursor on the graph of the line. Move this cursor to the left of the x-intercept, hit **ENTER**. Now it says "right bound?" Move the cursor to the right of the x-intercept, hit **ENTER**. Now it says "guess?" Move your cursor to the left somewhere in between the left and right bound near the x-intercept. Hit **ENTER**. At the bottom of your screen it will display the coordinates of the x-intercept or the "zero" to the y-value. Use this to find the x-intercept.*

Note: With linear/straight line functions the zero is not really a "guess," but it is necessary to enter a "guess" so it will search and find the exact x-intercept between your right and left boundaries. With other types of functions (more than one x-intercept), they may be irrational numbers so "guess" is more appropriate to give it the correct limits to find a very close approximation between the left and right boundaries.

51. $Y_1 = -8x + 6$

52. $Y_1 = 4x - 7$

53. $Y_1 = \frac{3x+5}{4}$ Round your answer to the nearest thousandth.

Extensions

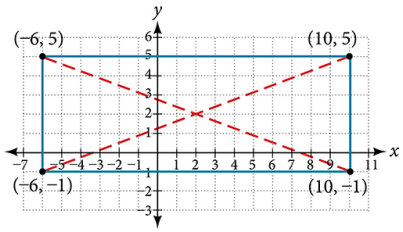
54. Someone drove 10 mi directly east from their home, made a left turn at an intersection, and then traveled 5 mi north to their place of work. If a road was made directly from the home to the place of work, what would its distance be to the nearest tenth of a mile?

55. If the road was made in the previous exercise, how much shorter would the person's one-way trip be every day?

56. Given these four points: $A(1, 3)$, $B(-3, 5)$, $C(4, 7)$, and $D(5, -4)$ find the coordinates of the midpoint of line segments $\overline{AB}$ and $\overline{CD}$.

57. After finding the two midpoints in the previous exercise, find the distance between the two midpoints to the nearest thousandth.

58. Given the graph of the rectangle shown and the coordinates of its vertices, prove that the diagonals of the rectangle are of equal length.



59. In the previous exercise, find the coordinates of the midpoint for each diagonal.

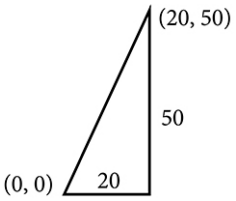
Real-World Applications

60. The coordinates on a map for San Francisco are $(53, 17)$ and those for Sacramento are $(128, 78)$. Note that coordinates represent miles. Find the distance between the cities to the nearest mile.

61. If San Jose's coordinates are $(76, -12)$, where the coordinates represent miles, find the distance between San Jose and San Francisco to the nearest mile.

62. A small craft in Lake Ontario sends out a distress signal. The coordinates of the boat in trouble were $(49, 64)$ One rescue boat is at the coordinates $(60, 82)$ and a second Coast Guard craft is at coordinates $(58, 47)$. Assuming both rescue craft travel at the same rate, which one would get to the distressed boat the fastest?

63. A person on the top of a building wants to have a guy wire extend to a point on the ground 20 ft from the building. To the nearest foot, how long will the wire have to be if the building is 50 ft tall?



64. If we rent a truck and pay a \$75/day fee plus \$.20 for every mile we travel, write a linear equation that would express the total cost per day y, using x to represent the number of miles we travel. Graph this function on your graphing calculator and find the total cost for one day if we travel 70 mi.

2.2 Linear Equations in One Variable

Learning Objectives

In this section, you will:

- Solve equations in one variable algebraically.
- Solve a rational equation.
- Find a linear equation.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has taken a part-time job at the local bank that pays \$15.00/hr, and she opened a savings account with an initial deposit of \$400 on January 15. She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately \$2,500, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others, which generate graphs like the line in Figure 1.

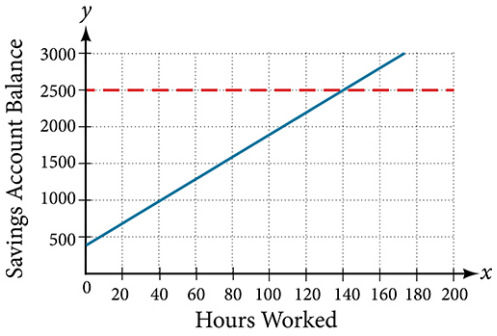


Figure 1

Solving Linear Equations in One Variable

A **linear equation** is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax + b = 0$ and are solved using basic algebraic operations.

We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent. An **identity equation** is true for all values of the variable. Here is an example of an identity equation.

$$3x = 2x + x$$

The **solution set** consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.

A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x + 2 = 3x - 6$, we have the following:

$$\begin{aligned} 5x + 2 &= 3x - 6 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

The solution set consists of one number: $\{-4\}$. It is the only solution and, therefore, we have solved a conditional equation.

An **inconsistent equation** results in a false statement. For example, if we are to solve $5x - 15 = 5(x - 4)$, we have the following:

$$\begin{aligned} 5x - 15 &= 5x - 20 \\ 5x - 15 - 5x &= 5x - 20 - 5x && \text{Subtract } 5x \text{ from both sides.} \\ -15 &\neq -20 && \text{False statement} \end{aligned}$$

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

Linear Equation in One Variable

A linear equation in one variable can be written in the form

$$ax + b = 0$$

where a and b are real numbers, $a \neq 0$.



HOW TO

Given a linear equation in one variable, use algebra to solve it.

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x =$_____, if x is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a(b+c) = ab + ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

EXAMPLE 1

Solving an Equation in One Variable

Solve the following equation: $2x + 7 = 19$.

Solution

This equation can be written in the form $ax + b = 0$ by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

$$\begin{aligned} 2x + 7 &= 19 & \\ 2x &= 12 & \text{Subtract 7 from both sides.} \\ x &= 6 & \text{Multiply both sides by } \tfrac{1}{2} \text{ or divide by 2.} \end{aligned}$$

The solution is 6.

TRY IT #1 Solve the linear equation in one variable: $2x + 1 = -9$.

EXAMPLE 2

Solving an Equation Algebraically When the Variable Appears on Both Sides

Solve the following equation: $4(x-3) + 12 = 15 - 5(x+6)$.

Solution

Apply standard algebraic properties.

$$\begin{aligned} 4(x-3) + 12 &= 15 - 5(x+6) & \\ 4x - 12 + 12 &= 15 - 5x - 30 & \text{Apply the distributive property.} \\ 4x &= -15 - 5x & \text{Combine like terms.} \\ 9x &= -15 & \text{Place } x\text{-terms on one side and simplify.} \\ x &= -\tfrac{15}{9} & \text{Multiply both sides by } \tfrac{1}{9}\text{, the reciprocal of 9.} \\ x &= -\tfrac{5}{3} & \end{aligned}$$

Analysis

This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x = -\frac{5}{3}$.

TRY IT #2 Solve the equation in one variable: $-2(3x-1) + x = 14 - x$.

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is a considered a **rational equation**.

Recall that a rational number is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{x+1}{x^2-4},\quad \frac{1}{x-3},\quad \text{or}\quad \frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the least common denominator (LCD).

Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

EXAMPLE 3

Solving a Rational Equation

Solve the rational equation: $\frac{7}{2x}-\frac{5}{3x}=\frac{22}{3}$.

Solution

We have three denominators; $2x$, $3x$, and 3. The LCD must contain $2x$, $3x$, and 3. An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.

$$\begin{aligned}
(6x)\left(\tfrac{7}{2x}-\tfrac{5}{3x}\right) &= \left(\tfrac{22}{3}\right)(6x) \\
(6x)\left(\tfrac{7}{2x}\right)-(6x)\left(\tfrac{5}{3x}\right) &= \left(\tfrac{22}{3}\right)(6x) && \text{Use the distributive property.} \\
(\cancel{6x})\left(\tfrac{7}{\cancel{2x}}\right)-(\cancel{6x})\left(\tfrac{5}{\cancel{3x}}\right) &= \left(\tfrac{22}{\cancel{3}}\right)(\cancel{6}\,x) && \text{Cancel out the common factors.} \\
3(7)-2(5) &= 22(2x) && \text{Multiply remaining factors by each numerator.} \\
21-10 &= 44x \\
11 &= 44x \\
\tfrac{11}{44} &= x \\
\tfrac{1}{4} &= x
\end{aligned}$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x, $x-1$, and $3x-3$. First, factor all denominators. We then have x, $(x-1)$, and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the x in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x, one factor of $(x-1)$, and the 3. Thus, the LCD is the following:

$$x\,(x-1)\,3=3x\,(x-1)$$

So, both sides of the equation would be multiplied by $3x\,(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and x^2+2x. Once the second denominator is factored as $x^2+2x=x\,(x+2)$, there is a common factor of x in both denominators and the LCD is $x\,(x+2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b}=\frac{c}{d}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{b}\times\frac{c}{d}.$$

Multiply $a\,(d)$ and $b\,(c)$, which results in $ad = bc$.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

Rational Equations

A **rational equation** contains at least one rational expression where the variable appears in at least one of the denominators.



HOW TO

Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator.

EXAMPLE 4

Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$$

Solution

We have three denominators: x, 2, and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, 0. Next, multiply the whole equation (both sides of the equal sign) by $2x$.

$$\begin{aligned} 2x\left(\tfrac{2}{x} - \tfrac{3}{2}\right) &= \left(\tfrac{7}{2x}\right)2x \\ 2\cancel{x}\left(\tfrac{2}{\cancel{x}}\right) - \cancel{2}x\left(\tfrac{3}{\cancel{2}}\right) &= \left(\tfrac{7}{\cancel{2x}}\right)\cancel{2x} && \text{Distribute } 2x. \\ 2(2) - 3x &= 7 && \text{Denominators cancel out.} \\ 4 - 3x &= 7 \\ -3x &= 3 \\ x &= -1 \\ &\text{or } \{-1\} \end{aligned}$$

The proposed solution is −1, which is not an excluded value, so the solution set contains one number, -1, or $\{-1\}$ written in set notation.

TRY IT #3 Solve the rational equation: $\frac{2}{3x} = \frac{1}{4} - \frac{1}{6x}$.

EXAMPLE 5

Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation: $\frac{1}{x} = \frac{1}{10} - \frac{3}{4x}$.

Solution

First find the common denominator. The three denominators in factored form are x, $10 = 2 \cdot 5$, and $4x = 2 \cdot 2 \cdot x$. The smallest expression that is divisible by each one of the denominators is $20x$. Only $x = 0$ is an excluded value. Multiply the whole equation by $20x$.

$$\begin{aligned} 20x\left(\tfrac{1}{x}\right) &= \left(\tfrac{1}{10} - \tfrac{3}{4x}\right)20x \\ 20 &= 2x - 15 \\ 35 &= 2x \\ \tfrac{35}{2} &= x \end{aligned}$$

The solution is $\frac{35}{2}$.

TRY IT #4 Solve the rational equation: $-\frac{5}{2x} + \frac{3}{4x} = -\frac{7}{4}$.

EXAMPLE 6

Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equations and state the excluded values:

ⓐ $\frac{3}{x-6} = \frac{5}{x}$ ⓑ $\frac{x}{x-3} = \frac{5}{x-3} - \frac{1}{2}$ ⓒ $\frac{x}{x-2} = \frac{5}{x-2} - \frac{1}{2}$

Solution

ⓐ

The denominators x and $x - 6$ have nothing in common. Therefore, the LCD is the product $x(x-6)$. However, for this problem, we can cross-multiply.

$$\begin{aligned} \tfrac{3}{x-6} &= \tfrac{5}{x} \\ 3x &= 5(x-6) \qquad \text{Distribute.} \\ 3x &= 5x - 30 \\ -2x &= -30 \\ x &= 15 \end{aligned}$$

The solution is 15. The excluded values are 6 and 0.

ⓑ

The LCD is $2(x-3)$. Multiply both sides of the equation by $2(x-3)$.

$$\begin{aligned} 2(x-3)\left(\tfrac{x}{x-3}\right) &= \left(\tfrac{5}{x-3} - \tfrac{1}{2}\right)2(x-3) \\ \frac{2\cancel{(x-3)}\,x}{\cancel{x-3}} &= \frac{2\cancel{(x-3)}\,5}{\cancel{x-3}} - \frac{\cancel{2}(x-3)}{\cancel{2}} \\ 2x &= 10 - (x-3) \\ 2x &= 10 - x + 3 \\ 2x &= 13 - x \\ 3x &= 13 \\ x &= \tfrac{13}{3} \end{aligned}$$

The solution is $\frac{13}{3}$. The excluded value is 3.

ⓒ
The least common denominator is $2(x-2)$. Multiply both sides of the equation by $2(x-2)$.

$$\begin{aligned} 2(x-2)\left(\tfrac{x}{x-2}\right) &= \left(\tfrac{5}{x-2}-\tfrac{1}{2}\right)2(x-2) \\ 2x &= 10-(x-2) \\ 2x &= 12-x \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

The solution is 4. The excluded value is 2.

TRY IT #5 Solve $\frac{-3}{2x+1}=\frac{4}{3x+1}$. State the excluded values.

EXAMPLE 7

Solving a Rational Equation with Factored Denominators and Stating Excluded Values
Solve the rational equation after factoring the denominators: $\frac{2}{x+1}-\frac{1}{x-1}=\frac{2x}{x^2-1}$. State the excluded values.

Solution
We must factor the denominator x^2-1. We recognize this as the difference of squares, and factor it as $(x-1)(x+1)$. Thus, the LCD that contains each denominator is $(x-1)(x+1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.

$$\begin{aligned} (x-1)(x+1)\left(\tfrac{2}{x+1}-\tfrac{1}{x-1}\right) &= \left(\tfrac{2x}{(x-1)(x+1)}\right)(x-1)(x+1) \\ 2(x-1)-1(x+1) &= 2x \\ 2x-2-x-1 &= 2x \quad \text{Distribute the negative sign.} \\ -3-x &= 0 \\ -3 &= x \end{aligned}$$

The solution is -3. The excluded values are 1 and -1.

TRY IT #6 Solve the rational equation: $\frac{2}{x-2}+\frac{1}{x+1}=\frac{1}{x^2-x-2}$.

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the slope-intercept form, written as $y=mx+b$, where $m=\text{slope}$ and $b=y\text{-intercept}$. Let us begin with the slope.

The Slope of a Line

The **slope** of a line refers to the ratio of the vertical change in *y* over the horizontal change in *x* between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$m=\frac{y_2-y_1}{x_2-x_1}$$

If the slope is positive, the line slants upward to the right. If the slope is negative, the line slants downward to the right. As the absolute value of the slope increases, the line becomes steeper. Some examples are shown in Figure 2. The lines indicate the following slopes: $m=-3$, $m=2$, and $m=\frac{1}{3}$.

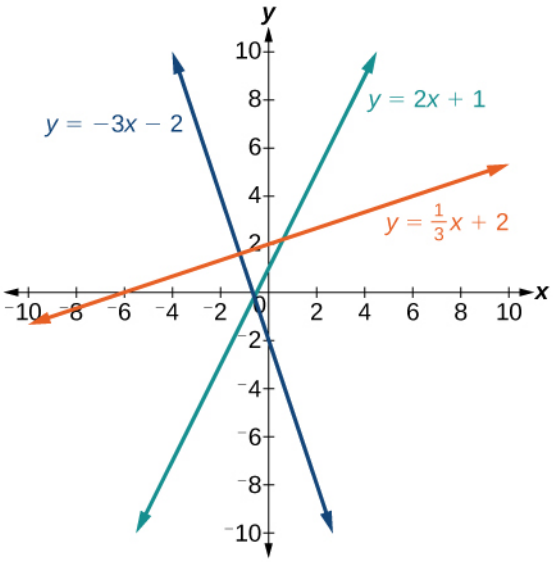


Figure 2

The Slope of a Line

The slope of a line, *m*, represents the change in *y* over the change in *x*. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE 8

Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2, -1)$ and $(-5, 3)$.

Solution

We substitute the *y*-values and the *x*-values into the formula.

$$\begin{aligned} m &= \frac{3-(-1)}{-5-2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$.

Analysis

It does not matter which point is called (x_1, y_1) or (x_2, y_2). As long as we are consistent with the order of the *y* terms and the order of the *x* terms in the numerator and denominator, the calculation will yield the same result.

TRY IT #7 Find the slope of the line that passes through the points $(-2, 6)$ and $(1, 4)$.

EXAMPLE 9

Identifying the Slope and *y*-intercept of a Line Given an Equation

Identify the slope and *y*-intercept, given the equation $y = -\frac{3}{4}x - 4$.

Solution

As the line is in $y = mx + b$ form, the given line has a slope of $m = -\frac{3}{4}$. The *y*-intercept is $b = -4$.

Analysis

The *y*-intercept is the point at which the line crosses the *y*-axis. On the *y*-axis, $x = 0$. We can always identify the *y*-intercept when the line is in slope-intercept form, as it will always equal *b*. Or, just substitute $x = 0$ and solve for *y*.

The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using the point-slope formula.

$$y - y_1 = m(x - x_1)$$

This is an important formula, as it will be used in other areas of college algebra and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

The Point-Slope Formula

Given one point and the slope, the point-slope formula will lead to the equation of a line:

$$y - y_1 = m(x - x_1)$$

EXAMPLE 10

Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope $m = -3$ and passing through the point $(4, 8)$. Write the final equation in slope-intercept form.

Solution

Using the point-slope formula, substitute -3 for *m* and the point $(4, 8)$ for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -3(x - 4) \\ y - 8 &= -3x + 12 \\ y &= -3x + 20 \end{aligned}$$

Analysis

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

TRY IT #8 Given $m = 4$, find the equation of the line in slope-intercept form passing through the point $(2, 5)$.

EXAMPLE 11

Finding the Equation of a Line Passing Through Two Given Points

Find the equation of the line passing through the points $(3, 4)$ and $(0, -3)$. Write the final equation in slope-intercept form.

Solution

First, we calculate the slope using the slope formula and two points.

$$\begin{aligned} m &= \frac{-3-4}{0-3} \\ &= \frac{-7}{-3} \\ &= \frac{7}{3} \end{aligned}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$, and either point. Let's pick the point $(3,4)$ for (x_1, y_1).

$$\begin{aligned} y-4 &= \tfrac{7}{3}(x-3) \\ y-4 &= \tfrac{7}{3}x-7 \qquad \text{Distribute the } \tfrac{7}{3}. \\ y &= \tfrac{7}{3}x-3 \end{aligned}$$

In slope-intercept form, the equation is written as $y=\frac{7}{3}x-3$.

Analysis

To prove that either point can be used, let us use the second point $(0,-3)$ and see if we get the same equation.

$$\begin{aligned} y-(-3) &= \tfrac{7}{3}(x-0) \\ y+3 &= \tfrac{7}{3}x \\ y &= \tfrac{7}{3}x-3 \end{aligned}$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Standard Form of a Line

Another way that we can represent the equation of a line is in standard form. Standard form is given as

$$Ax+By=C$$

where A, B, and C are integers. The *x*- and *y*-terms are on one side of the equal sign and the constant term is on the other side.

EXAMPLE 12

Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with $m=-6$ and passing through the point $\left(\frac{1}{4},-2\right)$. Write the equation in standard form.

Solution

We begin using the point-slope formula.

$$\begin{aligned} y-(-2) &= -6\left(x-\tfrac{1}{4}\right) \\ y+2 &= -6x+\tfrac{3}{2} \end{aligned}$$

From here, we multiply through by 2, as no fractions are permitted in standard form, and then move both variables to the left aside of the equal sign and move the constants to the right.

$$\begin{aligned} 2(y+2) &= \left(-6x+\tfrac{3}{2}\right)2 \\ 2y+4 &= -12x+3 \\ 12x+2y &= -1 \end{aligned}$$

This equation is now written in standard form.

TRY IT #9 Find the equation of the line in standard form with slope $m=-\frac{1}{3}$ and passing through the point $\left(1,\frac{1}{3}\right)$.

Vertical and Horizontal Lines

The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a vertical line is given as

$$x=c$$

where *c* is a constant. The slope of a vertical line is undefined, and regardless of the *y*-value of any point on the line, the *x*-coordinate of the point will be *c*.

Suppose that we want to find the equation of a line containing the following points: $(-3,-5)$, $(-3,1)$, $(-3,3)$, and $(-3,5)$. First, we will find the slope.

$$m=\frac{5-3}{-3-(-3)}=\frac{2}{0}$$

Zero in the denominator means that the slope is undefined and, therefore, we cannot use the point-slope formula. However, we can plot the points. Notice that all of the *x*-coordinates are the same and we find a vertical line through $x=-3$. See Figure 3.

The equation of a horizontal line is given as

$$y=c$$

where *c* is a constant. The slope of a horizontal line is zero, and for any *x*-value of a point on the line, the *y*-coordinate will be *c*.

Suppose we want to find the equation of a line that contains the following set of points: $(-2,-2)$, $(0,-2)$, $(3,-2)$, and $(5,-2)$. We can use the point-slope formula. First, we find the slope using any two points on the line.

$$\begin{aligned} m &= \frac{-2-(-2)}{0-(-2)} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Use any point for (x_1, y_1) in the formula, or use the *y*-intercept.

$$\begin{aligned} y-(-2) &= 0(x-3) \\ y+2 &= 0 \\ y &= -2 \end{aligned}$$

The graph is a horizontal line through $y=-2$. Notice that all of the *y*-coordinates are the same. See Figure 3.

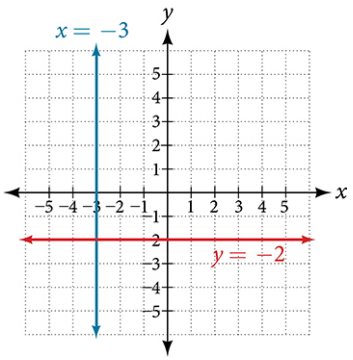


Figure 3 The line *x* = −3 is a vertical line. The line *y* = −2 is a horizontal line.

EXAMPLE 13

Finding the Equation of a Line Passing Through the Given Points

Find the equation of the line passing through the given points: $(1,-3)$ and $(1,4)$.

Solution

The *x*-coordinate of both points is 1. Therefore, we have a vertical line, $x=1$.

TRY IT #10 Find the equation of the line passing through $(-5,2)$ and $(2,2)$.

Determining Whether Graphs of Lines are Parallel or Perpendicular

Parallel lines have the same slope and different *y*-intercepts. Lines that are parallel to each other will never intersect. For example, Figure 4 shows the graphs of various lines with the same slope, $m=2$.

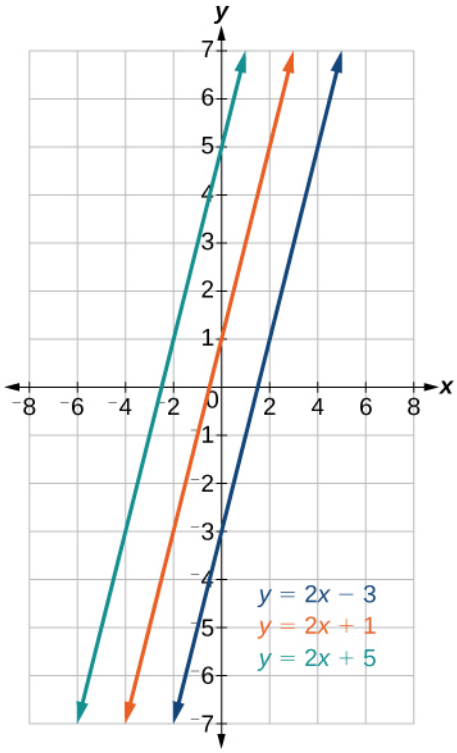


Figure 4 Parallel lines

All of the lines shown in the graph are parallel because they have the same slope and different *y*-intercepts.

Lines that are perpendicular intersect to form a $90°$ -angle. The slope of one line is the negative reciprocal of the other. We can show that two lines are perpendicular if the product of the two slopes is -1 : $m_1 \cdot m_2 = -1$. For example, Figure 5 shows the graph of two perpendicular lines. One line has a slope of 3; the other line has a slope of $-\frac{1}{3}$.

$$\begin{aligned} m_1 \cdot m_2 &= -1 \\ 3 \cdot \left(-\tfrac{1}{3}\right) &= -1 \end{aligned}$$

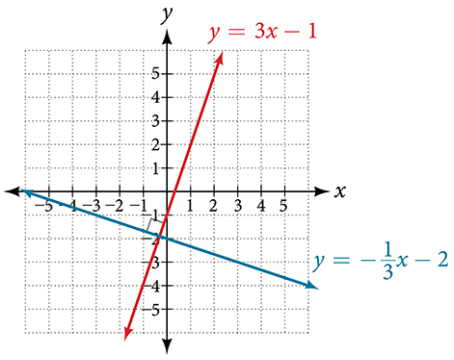


Figure 5 Perpendicular lines

EXAMPLE 14

Graphing Two Equations, and Determining Whether the Lines are Parallel, Perpendicular, or Neither

Graph the equations of the given lines, and state whether they are parallel, perpendicular, or neither: $3y = -4x + 3$ and $3x - 4y = 8$.

Solution

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.

First equation:

$$\begin{aligned} 3y &= -4x + 3 \\ y &= -\tfrac{4}{3}x + 1 \end{aligned}$$

Second equation:

$$\begin{aligned} 3x - 4y &= 8 \\ -4y &= -3x + 8 \\ y &= \tfrac{3}{4}x - 2 \end{aligned}$$

See the graph of both lines in Figure 6

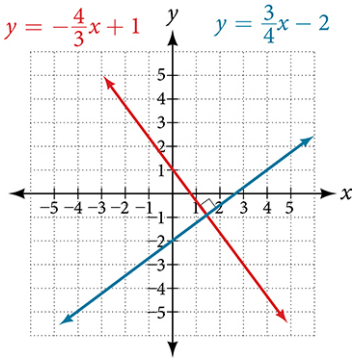


Figure 6

From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

$$\begin{aligned} m_1 &= -\tfrac{4}{3} \\ m_2 &= \tfrac{3}{4} \\ m_1 \cdot m_2 &= \left(-\tfrac{4}{3}\right)\left(\tfrac{3}{4}\right) = -1 \end{aligned}$$

The slopes are negative reciprocals of each other, confirming that the lines are perpendicular.

TRY IT #11 Graph the two lines and determine whether they are parallel, perpendicular, or neither: $2y - x = 10$ and $2y = x + 4$.

Writing the Equations of Lines Parallel or Perpendicular to a Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use the point-slope formula to write the equation of the new line.

HOW TO

Given an equation for a line, write the equation of a line parallel or perpendicular to it.

1. Find the slope of the given line. The easiest way to do this is to write the equation in slope-intercept form.
2. Use the slope and the given point with the point-slope formula.
3. Simplify the line to slope-intercept form and compare the equation to the given line.

EXAMPLE 15

Writing the Equation of a Line Parallel to a Given Line Passing Through a Given Point

Write the equation of line parallel to a $5x + 3y = 1$ and passing through the point $(3, 5)$.

Solution

First, we will write the equation in slope-intercept form to find the slope.

$$\begin{aligned} 5x+3y &= 1 \\ 3y &= -5x+1 \\ y &= -\tfrac{5}{3}x+\tfrac{1}{3} \end{aligned}$$

The slope is $m=-\frac{5}{3}$. The y-intercept is $\frac{1}{3}$, but that really does not enter into our problem, as the only thing we need for two lines to be parallel is the same slope. The one exception is that if the y-intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point with the point-slope formula.

$$\begin{aligned} y-5 &= -\tfrac{5}{3}(x-3) \\ y-5 &= -\tfrac{5}{3}x+5 \\ y &= -\tfrac{5}{3}x+10 \end{aligned}$$

The equation of the line is $y=-\frac{5}{3}x+10$. See Figure 7.

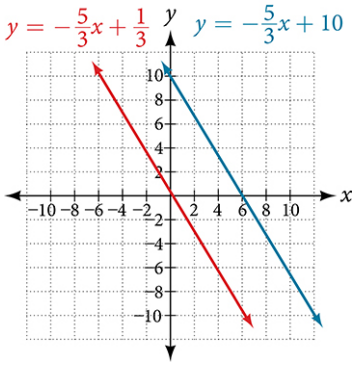


Figure 7

TRY IT #12 Find the equation of the line parallel to $5x=7+y$ and passing through the point $(-1,-2)$.

EXAMPLE 16

Finding the Equation of a Line Perpendicular to a Given Line Passing Through a Given Point

Find the equation of the line perpendicular to $5x-3y+4=0$ and passing through the point $(-4,1)$.

Solution

The first step is to write the equation in slope-intercept form.

$$\begin{aligned} 5x-3y+4 &= 0 \\ -3y &= -5x-4 \\ y &= \tfrac{5}{3}x+\tfrac{4}{3} \end{aligned}$$

We see that the slope is $m=\frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal, or $-\frac{3}{5}$. Next, we use the point-slope formula with this new slope and the given point.

$$\begin{aligned} y-1 &= -\tfrac{3}{5}(x-(-4)) \\ y-1 &= -\tfrac{3}{5}x-\tfrac{12}{5} \\ y &= -\tfrac{3}{5}x-\tfrac{12}{5}+\tfrac{5}{5} \\ y &= -\tfrac{3}{5}x-\tfrac{7}{5} \end{aligned}$$

MEDIA

Access these online resources for additional instruction and practice with linear equations.

Solving rational equations (https://openstax.org/l/rationaleqs)
Equation of a line given two points (https://openstax.org/l/twopointsline)
Finding the equation of a line perpendicular to another line through a given point (https://openstax.org/l/findperpline)
Finding the equation of a line parallel to another line through a given point (https://openstax.org/l/findparaline)

2.2 SECTION EXERCISES

Verbal

1. What does it mean when we say that two lines are parallel?

2. What is the relationship between the slopes of perpendicular lines (assuming neither is horizontal nor vertical)?

3. How do we recognize when an equation, for example $y = 4x + 3$, will be a straight line (linear) when graphed?

4. What does it mean when we say that a linear equation is inconsistent?

5. When solving the following equation: $\frac{2}{x-5} = \frac{4}{x+1}$ explain why we must exclude $x = 5$ and $x = -1$ as possible solutions from the solution set.

Algebraic

For the following exercises, solve the equation for x.

6. $7x + 2 = 3x - 9$

7. $4x - 3 = 5$

8. $3(x + 2) - 12 = 5(x + 1)$

9. $12 - 5(x + 3) = 2x - 5$

10. $\frac{1}{2} - \frac{1}{3}x = \frac{4}{3}$

11. $\frac{x}{3} - \frac{3}{4} = \frac{2x+3}{12}$

12. $\frac{2}{3}x + \frac{1}{2} = \frac{31}{6}$

13. $3(2x - 1) + x = 5x + 3$

14. $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$

15. $\frac{x+2}{4} - \frac{x-1}{3} = 2$

For the following exercises, solve each rational equation for x. State all x-values that are excluded from the solution set.

16. $\frac{3}{x} - \frac{1}{3} = \frac{1}{6}$

17. $2 - \frac{3}{x+4} = \frac{x+2}{x+4}$

18. $\frac{3}{x-2} = \frac{1}{x-1} + \frac{7}{(x-1)(x-2)}$

19. $\frac{3x}{x-1} + 2 = \frac{3}{x-1}$

20. $\frac{5}{x+1} + \frac{1}{x-3} = \frac{-6}{x^2-2x-3}$

21. $\frac{1}{x} = \frac{1}{5} + \frac{3}{2x}$

For the following exercises, find the equation of the line using the point-slope formula. Write all the final equations using the slope-intercept form.

22. $(0, 3)$ with a slope of $\frac{2}{3}$

23. $(1, 2)$ with a slope of $-\frac{4}{5}$

24. *x*-intercept is 1, and $(-2, 6)$

25. *y*-intercept is 2, and $(4, -1)$

26. $(-3, 10)$ and $(5, -6)$

27. $(1, 3)$ and $(5, 5)$

28. parallel to $y = 2x + 5$ and passes through the point $(4, 3)$

29. perpendicular to $3y = x - 4$ and passes through the point $(-2, 1)$.

For the following exercises, find the equation of the line using the given information.

30. $(-2, 0)$ and $(-2, 5)$

31. $(1, 7)$ and $(3, 7)$

32. The slope is undefined and it passes through the point $(2, 3)$.

33. The slope equals zero and it passes through the point $(1, -4)$.

34. The slope is $\frac{3}{4}$ and it passes through the point $(1, 4)$.

35. $(-1, 3)$ and $(4, -5)$

Graphical

For the following exercises, graph the pair of equations on the same axes, and state whether they are parallel, perpendicular, or neither.

36.
$$y = 2x + 7$$
$$y = -\frac{1}{2}x - 4$$

37.
$$3x - 2y = 5$$
$$6y - 9x = 6$$

38.
$$y = \frac{3x+1}{4}$$
$$y = 3x + 2$$

39.
$$x = 4$$
$$y = -3$$

Numeric

For the following exercises, find the slope of the line that passes through the given points.

40. $(5, 4)$ and $(7, 9)$

41. $(-3, 2)$ and $(4, -7)$

42. $(-5, 4)$ and $(2, 4)$

43. $(-1, -2)$ and $(3, 4)$

44. $(3, 3)$ and $(3, -2)$

For the following exercises, find the slope of the lines that pass through each pair of points and determine whether the lines are parallel or perpendicular.

45.
$$(-1, 3) \quad \text{and} \quad (5, 1)$$
$$(-2, 3) \quad \text{and} \quad (0, 9)$$

46.
$$(2, 5) \quad \text{and} \quad (5, 9)$$
$$(-1, -1) \quad \text{and} \quad (2, 3)$$

Technology

For the following exercises, express the equations in slope intercept form (rounding each number to the thousandths place). Enter this into a graphing calculator as Y1, then adjust the ymin and ymax values for your window to include where the y-intercept occurs. State your ymin and ymax values.

47. $0.537x - 2.19y = 100$

48. $4{,}500x - 200y = 9{,}528$

49. $\frac{200-30y}{x} = 70$

Extensions

50. Starting with the point-slope formula $y - y_1 = m(x - x_1)$, solve this expression for x in terms of x_1, y, y_1, and m.

51. Starting with the standard form of an equation $Ax + By = C$ solve this expression for y in terms of A, B, C and x. Then put the expression in slope-intercept form.

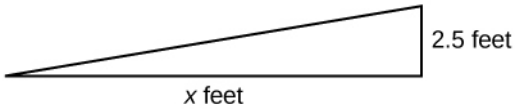
52. Use the above derived formula to put the following standard equation in slope intercept form: $7x - 5y = 25$.

53. Given that the following coordinates are the vertices of a rectangle, prove that this truly is a rectangle by showing the slopes of the sides that meet are perpendicular. $(-1, 1), (2, 0), (3, 3)$ and $(0, 4)$

54. Find the slopes of the diagonals in the previous exercise. Are they perpendicular?

Real-World Applications

55. The slope for a wheelchair ramp for a home has to be $\frac{1}{12}$. If the vertical distance from the ground to the door bottom is 2.5 ft, find the distance the ramp has to extend from the home in order to comply with the needed slope.



56. If the profit equation for a small business selling x number of item one and y number of item two is $p = 3x + 4y$, find the y value when $p = \$453$ and $x = 75$.

For the following exercises, use this scenario: The cost of renting a car is \$45/wk plus \$0.25/mi traveled during that week. An equation to represent the cost would be $y = 45 + .25x$, where x is the number of miles traveled.

57. What is your cost if you travel 50 mi?

58. If your cost were $\$63.75$, how many miles were you charged for traveling?

59. Suppose you have a maximum of \$100 to spend for the car rental. What would be the maximum number of miles you could travel?

2.3 Models and Applications

Learning Objectives

In this section, you will:

- Set up a linear equation to solve a real-world application.
- Use a formula to solve a real-world application.



Figure 1 Credit: Kevin Dooley

Neka is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum number of points that can be earned is 100. Is it possible for Neka to end the course with an A? A simple linear equation will give Neka his answer.

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce x widgets per month plus monthly operating charges; a car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this

section, we will set up and use linear equations to solve such problems.

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as \$0.10/mi, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10x$. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges \$0.10/mi plus a daily fee of \$50. We can use these quantities to model an equation that can be used to find the daily car rental cost C.

$$C = 0.10x + 50$$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. Table 1 lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by *a*	$x,\ \ x + a$
Twice a number	$2x$
One number is *a* more than another number	$x,\ \ x + a$
One number is *a* less than twice another number	$x,\ \ 2x - a$
The product of a number and *a*, decreased by *b*	$ax - b$
The quotient of a number and the number plus *a* is three times the number	$\frac{x}{x+a} = 3x$
The product of three times a number and the number decreased by *b* is *c*	$3x\,(x - b) = c$

Table 1



HOW TO

Given a real-world problem, model a linear equation to fit it.

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

EXAMPLE 1

Modeling a Linear Equation to Solve an Unknown Number Problem

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

Let x equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x + 17$. The sum of the two numbers is 31. We usually interpret the word *is* as an equal sign.

$$\begin{aligned} x + (x + 17) &= 31 \\ 2x + 17 &= 31 \quad \text{Simplify and solve.} \\ 2x &= 14 \\ x &= 7 \\ x + 17 &= 7 + 17 \\ &= 24 \end{aligned}$$

The two numbers are 7 and 24.

TRY IT #1 Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is 36, find the numbers.

EXAMPLE 2

Setting Up a Linear Equation to Solve a Real-World Application

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of $34 plus $.05/min talk-time. Company B charges a monthly service fee of $40 plus $.04/min talk-time.

(a) Write a linear equation that models the packages offered by both companies.
(b) If the average number of minutes used each month is 1,160, which company offers the better plan?
(c) If the average number of minutes used each month is 420, which company offers the better plan?
(d) How many minutes of talk-time would yield equal monthly statements from both companies?

Solution

(a) The model for Company *A* can be written as $A = 0.05x + 34$. This includes the variable cost of $0.05x$ plus the monthly service charge of $34. Company *B*'s package charges a higher monthly fee of $40, but a lower variable cost of $0.04x$. Company *B*'s model can be written as $B = 0.04x + \$40$.

(b)
If the average number of minutes used each month is 1,160, we have the following:

$$\begin{aligned} \text{Company } A &= 0.05(1,160) + 34 \\ &= 58 + 34 \\ &= 92 \\ \text{Company } B &= 0.04(1,160) + 40 \\ &= 46.4 + 40 \\ &= 86.4 \end{aligned}$$

So, Company *B* offers the lower monthly cost of $86.40 as compared with the $92 monthly cost offered by Company *A* when the average number of minutes used each month is 1,160.

(c)
If the average number of minutes used each month is 420, we have the following:

$$\begin{aligned} \text{Company } A &= 0.05(420) + 34 \\ &= 21 + 34 \\ &= 55 \\ \text{Company } B &= 0.04(420) + 40 \\ &= 16.8 + 40 \\ &= 56.8 \end{aligned}$$

If the average number of minutes used each month is 420, then Company *A* offers a lower monthly cost of $55 compared to Company *B*'s monthly cost of $56.80.

ⓓ

To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of (x, y) coordinates: At what point are both the *x*-value and the *y*-value equal? We can find this point by setting the equations equal to each other and solving for *x*.

$$\begin{aligned} 0.05x + 34 &= 0.04x + 40 \\ 0.01x &= 6 \\ x &= 600 \end{aligned}$$

Check the *x*-value in each equation.

$$\begin{aligned} 0.05(600) + 34 &= 64 \\ 0.04(600) + 40 &= 64 \end{aligned}$$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal. See Figure 2

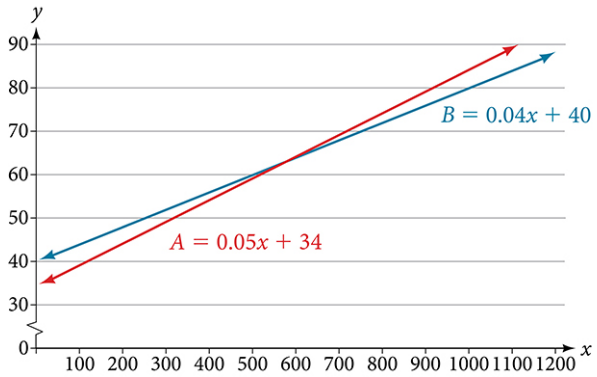


Figure 2

TRY IT #2 Find a linear equation to model this real-world application: It costs ABC electronics company $2.50 per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of $350 for utilities and $3,300 for salaries. What are the company's monthly expenses?

Using a Formula to Solve a Real-World Application

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulas include the **area** of a rectangular region, $A = LW$; the **perimeter** of a rectangle, $P = 2L + 2W$; and the **volume** of a rectangular solid, $V = LWH$. When there are two unknowns, we find a way to write one in terms of the other because we can solve for only one variable at a time.

EXAMPLE 3

Solving an Application Using a Formula

It takes Andrew 30 min to drive to work in the morning. He drives home using the same route, but it takes 10 min longer, and he averages 10 mi/h less than in the morning. How far does Andrew drive to work?

Solution

This is a distance problem, so we can use the formula $d = rt$, where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes 30 min, or $\frac{1}{2}$ h at rate *r*. His drive home takes 40 min, or $\frac{2}{3}$ h, and his speed averages 10 mi/h less than the morning drive. Both trips cover distance

d. A table, such as Table 2, is often helpful for keeping track of information in these types of problems.

	d	r	t
To Work	d	r	$\frac{1}{2}$
To Home	d	$r-10$	$\frac{2}{3}$

Table 2

Write two equations, one for each trip.

$$\begin{aligned} d &= r\left(\tfrac{1}{2}\right) && \text{To work} \\ d &= (r-10)\left(\tfrac{2}{3}\right) && \text{To home} \end{aligned}$$

As both equations equal the same distance, we set them equal to each other and solve for r.

$$\begin{aligned} r\left(\tfrac{1}{2}\right) &= (r-10)\left(\tfrac{2}{3}\right) \\ \tfrac{1}{2}r &= \tfrac{2}{3}r - \tfrac{20}{3} \\ \tfrac{1}{2}r - \tfrac{2}{3}r &= -\tfrac{20}{3} \\ -\tfrac{1}{6}r &= -\tfrac{20}{3} \\ r &= -\tfrac{20}{3}(-6) \\ r &= 40 \end{aligned}$$

We have solved for the rate of speed to work, 40 mph. Substituting 40 into the rate on the return trip yields 30 mi/h. Now we can answer the question. Substitute the rate back into either equation and solve for d.

$$\begin{aligned} d &= 40\left(\tfrac{1}{2}\right) \\ &= 20 \end{aligned}$$

The distance between home and work is 20 mi.

Analysis

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for r.

$$\begin{aligned} r\left(\tfrac{1}{2}\right) &= (r-10)\left(\tfrac{2}{3}\right) \\ 6 \times r\left(\tfrac{1}{2}\right) &= 6 \times (r-10)\left(\tfrac{2}{3}\right) \\ 3r &= 4(r-10) \\ 3r &= 4r-40 \\ -r &= -40 \\ r &= 40 \end{aligned}$$

TRY IT #3 On Saturday morning, it took Jennifer 3.6 h to drive to her mother's house for the weekend. On Sunday evening, due to heavy traffic, it took Jennifer 4 h to return home. Her speed was 5 mi/h slower on Sunday than on Saturday. What was her speed on Sunday?

EXAMPLE 4

Solving a Perimeter Problem

The perimeter of a rectangular outdoor patio is 54 ft. The length is 3 ft greater than the width. What are the dimensions of the patio?

Solution

The perimeter formula is standard: $P = 2L + 2W$. We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L = W + 3$. Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides as in Figure 3.

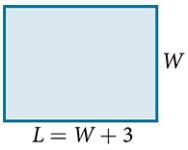


Figure 3

Now we can solve for the width and then calculate the length.

$$\begin{aligned} P &= 2L + 2W \\ 54 &= 2(W + 3) + 2W \\ 54 &= 2W + 6 + 2W \\ 54 &= 4W + 6 \\ 48 &= 4W \\ 12 &= W \\ (12 + 3) &= L \\ 15 &= L \end{aligned}$$

The dimensions are $L = 15$ ft and $W = 12$ ft.

TRY IT #4 Find the dimensions of a rectangle given that the perimeter is 110 cm and the length is 1 cm more than twice the width.

EXAMPLE 5

Solving an Area Problem

The perimeter of a tablet of graph paper is 48 in. The length is 6 in. more than the width. Find the area of the graph paper.

Solution

The standard formula for area is $A = LW$; however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is 6 in. more than the width, so we can write length as $L = W + 6$. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$$\begin{aligned} P &= 2L + 2W \\ 48 &= 2(W + 6) + 2W \\ 48 &= 2W + 12 + 2W \\ 48 &= 4W + 12 \\ 36 &= 4W \\ 9 &= W \\ (9 + 6) &= L \\ 15 &= L \end{aligned}$$

Now, we find the area given the dimensions of $L = 15$ in. and $W = 9$ in.

$$\begin{aligned} A &= LW \\ A &= 15(9) \\ &= 135 \text{ in.}^2 \end{aligned}$$

The area is 135 in.^{2}.

TRY IT #5 A game room has a perimeter of 70 ft. The length is five more than twice the width. How many ft^2 of new carpeting should be ordered?

EXAMPLE 6

Solving a Volume Problem

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is 1,600 in.^{3}.

Solution

The formula for the volume of a box is given as $V = LWH$, the product of length, width, and height. We are given that $L = 2W$, and $H = 8$. The volume is $1{,}600$ cubic inches.

$$\begin{aligned} V &= LWH \\ 1{,}600 &= (2W)W(8) \\ 1{,}600 &= 16W^2 \\ 100 &= W^2 \\ 10 &= W \end{aligned}$$

The dimensions are $L = 20$ in., $W = 10$ in., and $H = 8$ in.

Analysis

Note that the square root of W^2 would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

MEDIA

Access these online resources for additional instruction and practice with models and applications of linear equations.

Problem solving using linear equations (https://openstax.org/l/lineqprobsolve)
Problem solving using equations (https://openstax.org/l/equationprsolve)
Finding the dimensions of area given the perimeter (https://openstax.org/l/permareasolve)
Find the distance between the cities using the distance = rate * time formula (https://openstax.org/l/ratetimesolve)
Linear equation application (Write a cost equation) (https://openstax.org/l/lineqappl)

2.3 SECTION EXERCISES

Verbal

1. To set up a model linear equation to fit real-world applications, what should always be the first step?

2. Use your own words to describe this equation where n is a number: $5(n + 3) = 2n$

3. If the total amount of money you had to invest was $2,000 and you deposit x amount in one investment, how can you represent the remaining amount?

4. If a carpenter sawed a 10-ft board into two sections and one section was n ft long, how long would the other section be in terms of n ?

5. If Bill was traveling v mi/h, how would you represent Daemon's speed if he was traveling 10 mi/h faster?

Real-World Applications

For the following exercises, use the information to find a linear algebraic equation model to use to answer the question being asked.

6. Mark and Don are planning to sell each of their marble collections at a garage sale. If Don has 1 more than 3 times the number of marbles Mark has, how many does each boy have to sell if the total number of marbles is 113?

7. Beth and Ann are joking that their combined ages equal Sam's age. If Beth is twice Ann's age and Sam is 69 yr old, what are Beth and Ann's ages?

8. Ruden originally filled out 8 more applications than Hanh. Then each boy filled out 3 additional applications, bringing the total to 28. How many applications did each boy originally fill out?

For the following exercises, use this scenario: Two different telephone carriers offer the following plans that a person is considering. Company A has a monthly fee of $20 and charges of $.05/min for calls. Company B has a monthly fee of $5 and charges $.10/min for calls.

9. Find the model of the total cost of Company A's plan, using m for the minutes.

10. Find the model of the total cost of Company B's plan, using m for the minutes.

11. Find out how many minutes of calling would make the two plans equal.

12. If the person makes a monthly average of 200 min of calls, which plan should for the person choose?

For the following exercises, use this scenario: A wireless carrier offers the following plans that a person is considering. The Family Plan: $90 monthly fee, unlimited talk and text on up to 8 lines, and data charges of $40 for each device for up to 2 GB of data per device. The Mobile Share Plan: $120 monthly fee for up to 10 devices, unlimited talk and text for all the lines, and data charges of $35 for each device up to a shared total of 10 GB of data. Use P for the number of devices that need data plans as part of their cost.

13. Find the model of the total cost of the Family Plan.

14. Find the model of the total cost of the Mobile Share Plan.

15. Assuming they stay under their data limit, find the number of devices that would make the two plans equal in cost.

16. If a family has 3 smart phones, which plan should they choose?

For exercises 17 and 18, use this scenario: A retired woman has $50,000 to invest but needs to make $6,000 a year from the interest to meet certain living expenses. One bond investment pays 15% annual interest. The rest of it she wants to put in a CD that pays 7%.

17. If we let x be the amount the woman invests in the 15% bond, how much will she be able to invest in the CD?

18. Set up and solve the equation for how much the woman should invest in each option to sustain a $6,000 annual return.

19. Two planes fly in opposite directions. One travels 450 mi/h and the other 550 mi/h. How long will it take before they are 4,000 mi apart?

20. Ben starts walking along a path at 4 mi/h. One and a half hours after Ben leaves, his sister Amanda begins jogging along the same path at 6 mi/h. How long will it be before Amanda catches up to Ben?

21. Fiora starts riding her bike at 20 mi/h. After a while, she slows down to 12 mi/h, and maintains that speed for the rest of the trip. The whole trip of 70 mi takes her 4.5 h. For what distance did she travel at 20 mi/h?

22. A chemistry teacher needs to mix a 30% salt solution with a 70% salt solution to make 20 qt of a 40% salt solution. How many quarts of each solution should the teacher mix to get the desired result?

23. Raúl has $20,000 to invest. His intent is to earn 11% interest on his investment. He can invest part of his money at 8% interest and part at 12% interest. How much does Raúl need to invest in each option to make get a total 11% return on his $20,000?

For the following exercises, use this scenario: A truck rental agency offers two kinds of plans. Plan A charges $75/wk plus $.10/mi driven. Plan B charges $100/wk plus $.05/mi driven.

24. Write the model equation for the cost of renting a truck with plan A.

25. Write the model equation for the cost of renting a truck with plan B.

26. Find the number of miles that would generate the same cost for both plans.

27. If Tim knows he has to travel 300 mi, which plan should he choose?

For the following exercises, use the formula given to solve for the required value.

28. $A = P(1 + rt)$ is used to find the principal amount *P* deposited, earning *r*% interest, for *t* years. Use this to find what principal amount *P* David invested at a 3% rate for 20 yr if $A = \$8{,}000$.

29. The formula $F = \frac{mv^2}{R}$ relates force (F), velocity (v), mass , and resistance (m). Find R when $m = 45$, $v = 7$, and $F = 245$.

30. $F = ma$ indicates that force (*F*) equals mass (*m*) times acceleration (*a*). Find the acceleration of a mass of 50 kg if a force of 12 N is exerted on it.

31. $Sum = \frac{1}{1-r}$ is the formula for an infinite series sum. If the sum is 5, find *r*.

For the following exercises, solve for the given variable in the formula. After obtaining a new version of the formula, you will use it to solve a question.

32. Solve for *W*: $P = 2L + 2W$

33. Use the formula from the previous question to find the width, W, of a rectangle whose length is 15 and whose perimeter is 58.

34. Solve for f : $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$

35. Use the formula from the previous question to find f when $p = 8 \text{and } q = 13$.

36. Solve for m in the slope-intercept formula: $y = mx + b$

37. Use the formula from the previous question to find m when the coordinates of the point are $(4, 7)$ and $b = 12$.

38. The area of a trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2)$. Use the formula to find the area of a trapezoid with $h = 6, \ \ b_1 = 14, \ \text{and } b_2 = 8$.

39. Solve for *h:* $A = \frac{1}{2}h(b_1 + b_2)$

40. Use the formula from the previous question to find the height of a trapezoid with $A = 150, b_1 = 19,$ and $b_2 = 11$.

41. Find the dimensions of an American football field. The length is 200 ft more than the width, and the perimeter is 1,040 ft. Find the length and width. Use the perimeter formula $P = 2L + 2W$.

42. Distance equals rate times time, $d = rt$. Find the distance Tom travels if he is moving at a rate of 55 mi/h for 3.5 h.

43. Using the formula in the previous exercise, find the distance that Susan travels if she is moving at a rate of 60 mi/h for 6.75 h.

44. What is the total distance that two people travel in 3 h if one of them is riding a bike at 15 mi/h and the other is walking at 3 mi/h?

45. If the area model for a triangle is $A = \frac{1}{2}bh$, find the area of a triangle with a height of 16 in. and a base of 11 in.

46. Solve for *h:* $A = \frac{1}{2}bh$

47. Use the formula from the previous question to find the height to the nearest tenth of a triangle with a base of 15 and an area of 215.

48. The volume formula for a cylinder is $V = \pi r^2 h$. Using the symbol π in your answer, find the volume of a cylinder with a radius, *r*, of 4 cm and a height of 14 cm.

49. Solve for *h:* $V = \pi r^2 h$

50. Use the formula from the previous question to find the height of a cylinder with a radius of 8 and a volume of 16π

51. Solve for *r:* $V = \pi r^2 h$

52. Use the formula from the previous question to find the radius of a cylinder with a height of 36 and a volume of 324π.

53. The formula for the circumference of a circle is $C = 2\pi r$. Find the circumference of a circle with a diameter of 12 in. (diameter = 2*r*). Use the symbol π in your final answer.

54. Solve the formula from the previous question for π. Notice why π is sometimes defined as the ratio of the circumference to its diameter.

2.4 Complex Numbers

Learning Objectives

In this section, you will:

- Add and subtract complex numbers.
- Multiply and divide complex numbers.
- Simplify powers of i

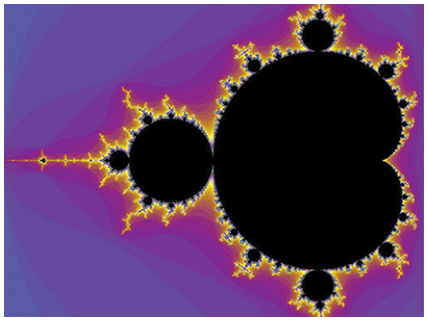


Figure 1

Discovered by Benoit Mandelbrot around 1980, the Mandelbrot Set is one of the most recognizable fractal images. The image is built on the theory of self-similarity and the operation of iteration. Zooming in on a fractal image brings many surprises, particularly in the high level of repetition of detail that appears as magnification increases. The equation that generates this image turns out to be rather simple.

In order to better understand it, we need to become familiar with a new set of numbers. Keep in mind that the study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. In this section, we will explore a set of numbers that fills voids in the set of real numbers and find out how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of i

We know how to find the square root of any positive real number. In a similar way, we can find the square root of any negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number**.** The imaginary number i is defined as the square root of -1.

$$\sqrt{-1} = i$$

So, using properties of radicals,

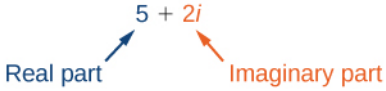
$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

We can write the square root of any negative number as a multiple of i. Consider the square root of -49.

$$\begin{aligned} \sqrt{-49} &= \sqrt{49 \cdot (-1)} \\ &= \sqrt{49}\sqrt{-1} \\ &= 7i \end{aligned}$$

We use $7i$ and not $-7i$ because the principal root of 49 is the positive root.

A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ where a is the real part and b is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4i\sqrt{3}$.



Imaginary numbers differ from real numbers in that a squared imaginary number produces a negative real number. Recall that when a positive real number is squared, the result is a positive real number and when a negative real number is squared, the result is also a positive real number. Complex numbers consist of real and imaginary numbers.

Imaginary and Complex Numbers

A **complex number** is a number of the form $a + bi$ where

- a is the real part of the complex number.
- b is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and b is not equal to 0, the complex number is called a pure imaginary number. An **imaginary number** is an even root of a negative number.



HOW TO

Given an imaginary number, express it in the standard form of a complex number.

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as i.
3. Write $\sqrt{a} \cdot i$ in simplest form.

EXAMPLE 1

Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

Solution

$$\begin{aligned}\sqrt{-9} &= \sqrt{9}\sqrt{-1} \\ &= 3i\end{aligned}$$

In standard form, this is $0 + 3i$.

TRY IT #1 Express $\sqrt{-24}$ in standard form.

Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number, we need to address the two components of the number. We use the **complex plane**, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs (a, b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

Let's consider the number $-2 + 3i$. The real part of the complex number is -2 and the imaginary part is 3. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$, as shown in Figure 2.

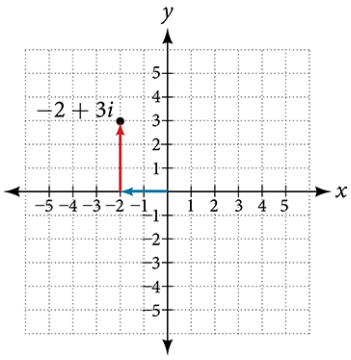


Figure 2

Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis, as shown in Figure 3.

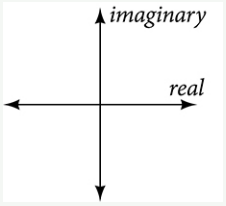


Figure 3

HOW TO

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

EXAMPLE 2

Plotting a Complex Number on the Complex Plane

Plot the complex number $3 - 4i$ on the complex plane.

Solution

The real part of the complex number is 3, and the imaginary part is –4. We plot the ordered pair $(3, -4)$ as shown in Figure 4.

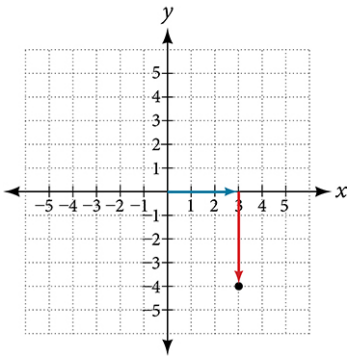


Figure 4

TRY IT #2 Plot the complex number $-4 - i$ on the complex plane.

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and then combine the imaginary parts.

Complex Numbers: Addition and Subtraction

Adding complex numbers:

$$(a+bi)+(c+di)=(a+c)+(b+d)\,i$$

Subtracting complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)\,i$$

HOW TO

Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

EXAMPLE 3

Adding and Subtracting Complex Numbers

Add or subtract as indicated.

ⓐ $(3-4i)+(2+5i)$ ⓑ $(-5+7i)-(-11+2i)$

Solution

We add the real parts and add the imaginary parts.

ⓐ
$$\begin{aligned}(3-4i)+(2+5i) &= 3-4i+2+5i\\ &= 3+2+(-4i)+5i\\ &= (3+2)+(-4+5)i\\ &= 5+i\end{aligned}$$

ⓑ
$$\begin{aligned}(-5+7i)-(-11+2i) &= -5+7i+11-2i\\ &= -5+11+7i-2i\\ &= (-5+11)+(7-2)i\\ &= 6+5i\end{aligned}$$

TRY IT #3 Subtract $2+5i$ from $3-4i$.

Multiplying Complex Numbers

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

Multiplying a Complex Number by a Real Number

Lets begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. Consider, for example, $3\,(6+2i)$:

$$\begin{aligned}3(6+2i) &= (3\cdot 6)+(3\cdot 2i) && \text{Distribute.}\\ &= 18+6i && \text{Simplify.}\end{aligned}$$

HOW TO

Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

EXAMPLE 4

Multiplying a Complex Number by a Real Number

Find the product $4(2+5i)$.

Solution

Distribute the 4.

$$\begin{aligned} 4(2+5i) &= (4\cdot 2)+(4\cdot 5i) \\ &= 8+20i \end{aligned}$$

TRY IT #4 Find the product: $\frac{1}{2}(5-2i)$.

Multiplying Complex Numbers Together

Now, let's multiply two complex numbers. We can use either the distributive property or more specifically the FOIL method because we are dealing with binomials. Recall that FOIL is an acronym for multiplying First, Inner, Outer, and Last terms together. The difference with complex numbers is that when we get a squared term, i^2, it equals -1.

$$\begin{aligned} (a+bi)(c+di) &= ac+adi+bci+bdi^2 & \\ &= ac+adi+bci-bd & i^2=-1 \\ &= (ac-bd)+(ad+bc)i & \text{Group real terms and imaginary terms.} \end{aligned}$$

HOW TO

Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. Remember that $i^2=-1$.
3. Group together the real terms and the imaginary terms

EXAMPLE 5

Multiplying a Complex Number by a Complex Number

Multiply: $(4+3i)(2-5i)$.

Solution

$$\begin{aligned} (4+3i)(2-5i) &= 4(2)-4(5i)+3i(2)-(3i)(5i) \\ &= 8-20i+6i-15\left(i^2\right) \\ &= (8+15)+(-20+6)i \\ &= 23-14i \end{aligned}$$

TRY IT #5 Multiply: $(3-4i)(2+3i)$.

Dividing Complex Numbers

Dividing two complex numbers is more complicated than adding, subtracting, or multiplying because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator to write the answer in standard form $a+bi$. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a+bi$ is $a-bi$. For example, the product of $a+bi$ and $a-bi$ is

$$\begin{aligned}(a+bi)(a-bi) &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 + b^2\end{aligned}$$

The result is a real number.

Note that complex conjugates have an opposite relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c+di$ by $a+bi$, where neither a nor b equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c+di}{a+bi} \text{ where } a \neq 0 \text{ and } b \neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)} \cdot \frac{(a-bi)}{(a-bi)} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$= \frac{ca - cbi + adi - bdi^2}{a^2 - abi + abi - b^2 i^2}$$

Simplify, remembering that $i^2 = -1$.

$$\begin{aligned}&= \frac{ca - cbi + adi - bd(-1)}{a^2 - abi + abi - b^2(-1)} \\ &= \frac{(ca+bd)+(ad-cb)i}{a^2+b^2}\end{aligned}$$

The Complex Conjugate

The **complex conjugate** of a complex number $a+bi$ is $a-bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

EXAMPLE 6

Finding Complex Conjugates

Find the complex conjugate of each number.

ⓐ $2 + i\sqrt{5}$ ⓑ $-\frac{1}{2}i$

Solution

ⓐ The number is already in the form $a+bi$. The complex conjugate is $a-bi$, or $2 - i\sqrt{5}$.

ⓑ We can rewrite this number in the form $a+bi$ as $0 - \frac{1}{2}i$. The complex conjugate is $a-bi$, or $0 + \frac{1}{2}i$. This can be written simply as $\frac{1}{2}i$.

Analysis

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by i.

TRY IT #6 Find the complex conjugate of $-3+4i$.

HOW TO

Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

EXAMPLE 7

Dividing Complex Numbers

Divide: $(2+5i)$ by $(4-i)$.

Solution

We begin by writing the problem as a fraction.

$$\frac{(2+5i)}{(4-i)}$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (using FOIL).

$$\begin{aligned} \frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} &= \frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \\ &= \frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} && \text{Because } i^2=-1. \\ &= \frac{3+22i}{17} \\ &= \frac{3}{17}+\frac{22}{17}i && \text{Separate real and imaginary parts.} \end{aligned}$$

Note that this expresses the quotient in standard form.

Simplifying Powers of *i*

The powers of i are cyclic. Let's look at what happens when we raise i to increasing powers.

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i \\ i^4 &= i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1 \\ i^5 &= i^4 \cdot i = 1 \cdot i = i \end{aligned}$$

We can see that when we get to the fifth power of i, it is equal to the first power. As we continue to multiply i by increasing powers, we will see a cycle of four. Let's examine the next four powers of i.

$$\begin{aligned} i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1 \\ i^7 &= i^6 \cdot i = i^2 \cdot i = i^3 = -i \\ i^8 &= i^7 \cdot i = i^3 \cdot i = i^4 = 1 \\ i^9 &= i^8 \cdot i = i^4 \cdot i = i^5 = i \end{aligned}$$

The cycle is repeated continuously: $i, -1, -i, 1$, every four powers.

EXAMPLE 8

Simplifying Powers of i
Evaluate: i^{35}.

Solution
Since $i^4 = 1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35 = 4 \cdot 8 + 3$.

$$i^{35} = i^{4 \cdot 8 + 3} = i^{4 \cdot 8} \cdot i^3 = \left(i^4\right)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i$$

TRY IT #7 Evaluate: i^{18}

Q&A **Can we write i^{35} in other helpful ways?**

As we saw in Example 8, we reduced i^{35} to i^3 by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of i^{35} may be more useful. Table 1 shows some other possible factorizations.

Factorization of i^{35}	$i^{34} \cdot i$	$i^{33} \cdot i^2$	$i^{31} \cdot i^4$	$i^{19} \cdot i^{16}$
Reduced form	$\left(i^2\right)^{17} \cdot i$	$i^{33} \cdot (-1)$	$i^{31} \cdot 1$	$i^{19} \cdot \left(i^4\right)^4$
Simplified form	$(-1)^{17} \cdot i$	$-i^{33}$	i^{31}	i^{19}

Table 1

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

MEDIA

Access these online resources for additional instruction and practice with complex numbers.

Adding and Subtracting Complex Numbers (https://openstax.org/l/addsubcomplex)
Multiply Complex Numbers (https://openstax.org/l/multiplycomplex)
Multiplying Complex Conjugates (https://openstax.org/l/multcompconj)
Raising *i* to Powers (https://openstax.org/l/raisingi)

2.4 SECTION EXERCISES

Verbal

1. Explain how to add complex numbers.

2. What is the basic principle in multiplication of complex numbers?

3. Give an example to show that the product of two imaginary numbers is not always imaginary.

4. What is a characteristic of the plot of a real number in the complex plane?

Algebraic

For the following exercises, evaluate the algebraic expressions.

5. If $y = x^2 + x - 4$, evaluate y given $x = 2i$.

6. If $y = x^3 - 2$, evaluate y given $x = i$.

7. If $y = x^2 + 3x + 5$, evaluate y given $x = 2 + i$.

8. If $y = 2x^2 + x - 3$, evaluate y given $x = 2 - 3i$.

9. If $y = \frac{x+1}{2-x}$, evaluate y given $x = 5i$.

10. If $y = \frac{1+2x}{x+3}$, evaluate y given $x = 4i$.

Graphical

For the following exercises, plot the complex numbers on the complex plane.

11. $1 - 2i$

12. $-2 + 3i$

13. i

14. $-3 - 4i$

Numeric

For the following exercises, perform the indicated operation and express the result as a simplified complex number.

15. $(3 + 2i) + (5 - 3i)$

16. $(-2 - 4i) + (1 + 6i)$

17. $(-5 + 3i) - (6 - i)$

18. $(2 - 3i) - (3 + 2i)$

19. $(-4 + 4i) - (-6 + 9i)$

20. $(2 + 3i)(4i)$

21. $(5 - 2i)(3i)$

22. $(6 - 2i)(5)$

23. $(-2 + 4i)(8)$

24. $(2 + 3i)(4 - i)$

25. $(-1 + 2i)(-2 + 3i)$

26. $(4 - 2i)(4 + 2i)$

27. $(3 + 4i)(3 - 4i)$

28. $\frac{3+4i}{2}$

29. $\frac{6-2i}{3}$

30. $\frac{-5+3i}{2i}$

31. $\frac{6+4i}{i}$

32. $\frac{2-3i}{4+3i}$

33. $\frac{3+4i}{2-i}$

34. $\frac{2+3i}{2-3i}$

35. $\sqrt{-9} + 3\sqrt{-16}$

36. $-\sqrt{-4} - 4\sqrt{-25}$

37. $\frac{2+\sqrt{-12}}{2}$

38. $\frac{4+\sqrt{-20}}{2}$

39. i^8

40. i^{15}

41. i^{22}

Technology

For the following exercises, use a calculator to help answer the questions.

42. Evaluate $(1 + i)^k$ for $k = 4, 8$, and 12. Predict the value if $k = 16$.

43. Evaluate $(1 - i)^k$ for $k = 2, 6$, and 10. Predict the value if $k = 14$.

44. Evaluate $(1 + i)^k - (1 - i)^k$ for $k = 4, 8$, and 12. Predict the value for $k = 16$.

45. Show that a solution of $x^6 + 1 = 0$ is $\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

46. Show that a solution of $x^8 - 1 = 0$ is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

Extensions

For the following exercises, evaluate the expressions, writing the result as a simplified complex number.

47. $\frac{1}{i} + \frac{4}{i^3}$

48. $\frac{1}{i^{11}} - \frac{1}{i^{21}}$

49. $i^7\left(1 + i^2\right)$

50. $i^{-3} + 5i^7$

51. $\frac{(2+i)(4-2i)}{(1+i)}$

52. $\frac{(1+3i)(2-4i)}{(1+2i)}$

53. $\frac{(3+i)^2}{(1+2i)^2}$

54. $\frac{3+2i}{2+i} + (4 + 3i)$

55. $\frac{4+i}{i} + \frac{3-4i}{1-i}$

56. $\frac{3+2i}{1+2i} - \frac{2-3i}{3+i}$

2.5 Quadratic Equations

Learning Objectives

In this section, you will:

- Solve quadratic equations by factoring.
- Solve quadratic equations by the square root property.
- Solve quadratic equations by completing the square.
- Solve quadratic equations by using the quadratic formula.



Figure 1

The computer monitor on the left in Figure 1 is a 23.6-inch model and the one on the right is a 27-inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.

Solving Quadratic Equations by Factoring

An equation containing a second-degree polynomial is called a quadratic equation. For example, equations such as $2x^2 + 3x - 1 = 0$ and $x^2 - 4 = 0$ are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is factoring. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a \cdot b = 0$, then $a = 0$ or $b = 0$, where a and b are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression $(x - 2)(x + 3)$ by multiplying the two factors together.

$$\begin{aligned}(x-2)(x+3) &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6\end{aligned}$$

The product is a quadratic expression. Set equal to zero, $x^2 + x - 6 = 0$ is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $ax^2 + bx + c = 0$, where a, b, and c are real numbers, and $a \neq 0$. The equation $x^2 + x - 6 = 0$ is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor (GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

The Zero-Product Property and Quadratic Equations

The **zero-product property** states

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0,$$

where a and b are real numbers or algebraic expressions.

A **quadratic equation** is an equation containing a second-degree polynomial; for example

$$ax^2 + bx + c = 0$$

where *a*, *b*, and *c* are real numbers, and if $a \neq 0$, it is in standard form.

Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^2 + x - 6 = 0$, the leading coefficient, or the coefficient of x^2, is 1. We have one method of factoring quadratic equations in this form.

HOW TO

Given a quadratic equation with the leading coefficient of 1, factor it.

1. Find two numbers whose product equals *c* and whose sum equals *b*.
2. Use those numbers to write two factors of the form $(x + k)$ or $(x - k)$, where *k* is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and -2, the factors are $(x + 1)(x - 2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

EXAMPLE 1

Factoring and Solving a Quadratic with Leading Coefficient of 1

Factor and solve the equation: $x^2 + x - 6 = 0$.

Solution

To factor $x^2 + x - 6 = 0$, we look for two numbers whose product equals -6 and whose sum equals 1. Begin by looking at the possible factors of -6.

$$1 \cdot (-6)$$
$$(-6) \cdot 1$$
$$2 \cdot (-3)$$
$$3 \cdot (-2)$$

The last pair, $3 \cdot (-2)$ sums to 1, so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$(x - 2)(x + 3) = 0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$\begin{aligned} (x - 2)(x + 3) &= 0 \\ (x - 2) &= 0 \\ x &= 2 \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The two solutions are 2 and -3. We can see how the solutions relate to the graph in Figure 2. The solutions are the *x*-intercepts of $y = x^2 + x - 6 = 0$.

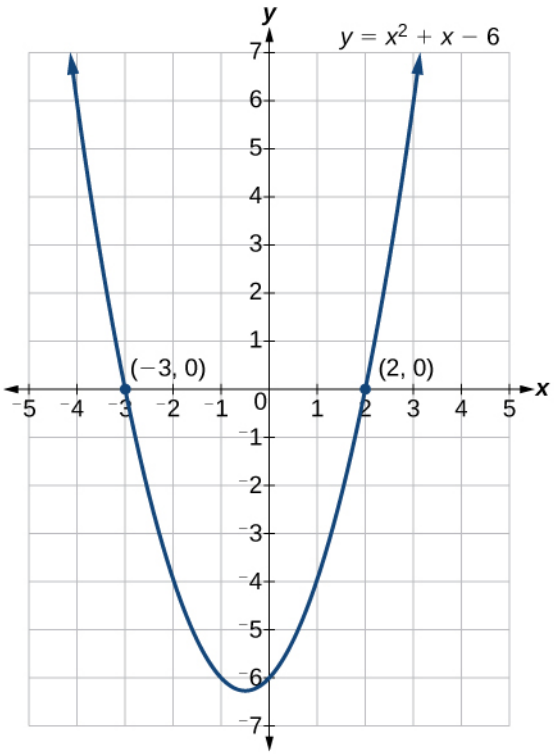


Figure 2

TRY IT #1 Factor and solve the quadratic equation: $x^2 - 5x - 6 = 0$.

EXAMPLE 2

Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring: $x^2 + 8x + 15 = 0$.

Solution

Find two numbers whose product equals 15 and whose sum equals 8. List the factors of 15.

$$\begin{aligned} &1 \cdot 15 \\ &3 \cdot 5 \\ &(-1) \cdot (-15) \\ &(-3) \cdot (-5) \end{aligned}$$

The numbers that add to 8 are 3 and 5. Then, write the factors, set each factor equal to zero, and solve.

$$\begin{aligned} (x+3)(x+5) &= 0 \\ (x+3) &= 0 \\ x &= -3 \\ (x+5) &= 0 \\ x &= -5 \end{aligned}$$

The solutions are -3 and -5.

TRY IT #2 Solve the quadratic equation by factoring: $x^2 - 4x - 21 = 0$.

EXAMPLE 3

Using the Zero-Product Property to Solve a Quadratic Equation Written as the Difference of Squares

Solve the difference of squares equation using the zero-product property: $x^2 - 9 = 0$.

Solution

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ (x-3) &= 0 \\ x &= 3 \\ (x+3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are 3 and -3.

TRY IT #3 Solve by factoring: $x^2 - 25 = 0$.

Solving a Quadratic Equation by Factoring when the Leading Coefficient is not 1

When the leading coefficient is not 1, we factor a quadratic equation using the method called grouping, which requires four terms. With the equation in standard form, let's review the grouping procedures:

1. With the quadratic in standard form, $ax^2 + bx + c = 0$, multiply $a \cdot c$.
2. Find two numbers whose product equals ac and whose sum equals b.
3. Rewrite the equation replacing the bx term with two terms using the numbers found in step 2 as coefficients of x.
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

EXAMPLE 4

Solving a Quadratic Equation Using Grouping

Use grouping to factor and solve the quadratic equation: $4x^2 + 15x + 9 = 0$.

Solution

First, multiply ac : $4\,(9) = 36$. Then list the factors of 36.

$$\begin{aligned} &1 \cdot 36 \\ &2 \cdot 18 \\ &3 \cdot 12 \\ &4 \cdot 9 \\ &6 \cdot 6 \end{aligned}$$

The only pair of factors that sums to 15 is $3 + 12$. Rewrite the equation replacing the b term, $15x$, with two terms using 3 and 12 as coefficients of x. Factor the first two terms, and then factor the last two terms.

$$\begin{aligned} 4x^2 + 3x + 12x + 9 &= 0 \\ x(4x+3) + 3(4x+3) &= 0 \\ (4x+3)(x+3) &= 0 \end{aligned}$$

Solve using the zero-product property.

$$\begin{aligned}(4x+3)(x+3) &= 0\\ (4x+3) &= 0\\ x &= -\tfrac{3}{4}\\ (x+3) &= 0\\ x &= -3\end{aligned}$$

The solutions are $-\frac{3}{4}$, and -3. See Figure 3.

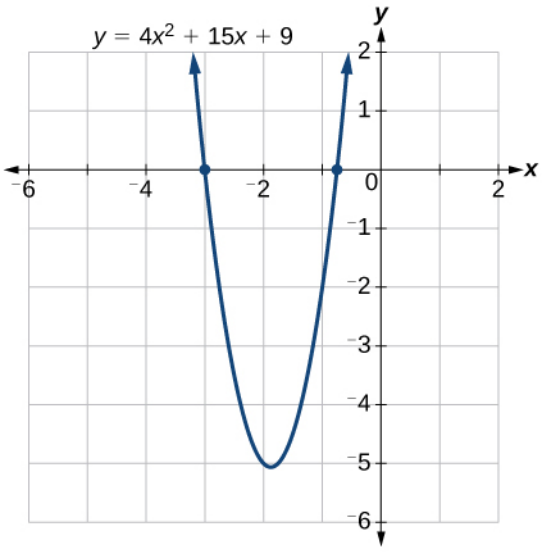


Figure 3

TRY IT #4 Solve using factoring by grouping: $12x^2 + 11x + 2 = 0$.

EXAMPLE 5

Solving a Polynomial of Higher Degree by Factoring

Solve the equation by factoring: $-3x^3 - 5x^2 - 2x = 0$.

Solution

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out $-x$ from all of the terms and then proceed with grouping.

$$\begin{aligned}-3x^3 - 5x^2 - 2x &= 0\\ -x\left(3x^2 + 5x + 2\right) &= 0\end{aligned}$$

Use grouping on the expression in parentheses.

$$\begin{aligned}-x\left(3x^2 + 3x + 2x + 2\right) &= 0\\ -x[3x(x+1) + 2(x+1)] &= 0\\ -x(3x+2)(x+1) &= 0\end{aligned}$$

Now, we use the zero-product property. Notice that we have three factors.

$$\begin{aligned} -x &= 0 \\ x &= 0 \\ 3x+2 &= 0 \\ x &= -\tfrac{2}{3} \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

The solutions are 0, $-\frac{2}{3}$, and -1.

TRY IT #5 Solve by factoring: $x^3 + 11x^2 + 10x = 0$.

Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the **square root property**, in which we isolate the x^2 term and take the square root of the number on the other side of the equals sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the x^2 term so that the square root property can be used.

The Square Root Property

With the x^2 term isolated, the square root property states that:

$$\text{if } x^2 = k, \text{then } x = \pm\sqrt{k}$$

where k is a nonzero real number.

HOW TO

Given a quadratic equation with an x^2 term but no x term, use the square root property to solve it.

1. Isolate the x^2 term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

EXAMPLE 6

Solving a Simple Quadratic Equation Using the Square Root Property

Solve the quadratic using the square root property: $x^2 = 8$.

Solution

Take the square root of both sides, and then simplify the radical. Remember to use a $\pm$ sign before the radical symbol.

$$\begin{aligned} x^2 &= 8 \\ x &= \pm\sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

The solutions are $2\sqrt{2}$, $-2\sqrt{2}$.

EXAMPLE 7

Solving a Quadratic Equation Using the Square Root Property

Solve the quadratic equation: $4x^2 + 1 = 7$.

Solution

First, isolate the x^2 term. Then take the square root of both sides.

$$\begin{aligned} 4x^2 + 1 &= 7 \\ 4x^2 &= 6 \\ x^2 &= \frac{6}{4} \\ x &= \pm\frac{\sqrt{6}}{2} \end{aligned}$$

The solutions are $\frac{\sqrt{6}}{2}$, and $-\frac{\sqrt{6}}{2}$.

TRY IT #6 Solve the quadratic equation using the square root property: $3(x-4)^2 = 15$.

Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a quadratic equation known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, *a*, must equal 1. If it does not, then divide the entire equation by *a*. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2 + 4x + 1 = 0$ to illustrate each step.

1. Given a quadratic equation that cannot be factored, and with $a = 1$, first add or subtract the constant term to the right side of the equal sign.
$$x^2 + 4x = -1$$
2. Multiply the *b* term by $\frac{1}{2}$ and square it.
$$\begin{aligned} \tfrac{1}{2}(4) &= 2 \\ 2^2 &= 4 \end{aligned}$$
3. Add $\left(\frac{1}{2}b\right)^2$ to both sides of the equal sign and simplify the right side. We have
$$\begin{aligned} x^2 + 4x + 4 &= -1 + 4 \\ x^2 + 4x + 4 &= 3 \end{aligned}$$
4. The left side of the equation can now be factored as a perfect square.
$$\begin{aligned} x^2 + 4x + 4 &= 3 \\ (x+2)^2 &= 3 \end{aligned}$$
5. Use the square root property and solve.
$$\begin{aligned} \sqrt{(x+2)^2} &= \pm\sqrt{3} \\ x + 2 &= \pm\sqrt{3} \\ x &= -2 \pm \sqrt{3} \end{aligned}$$
6. The solutions are $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$.

EXAMPLE 8

Solving a Quadratic by Completing the Square

Solve the quadratic equation by completing the square: $x^2 - 3x - 5 = 0$.

Solution

First, move the constant term to the right side of the equal sign.

$$x^2 - 3x = 5$$

Then, take $\frac{1}{2}$ of the b term and square it.

$$\begin{aligned} \tfrac{1}{2}(-3) &= -\tfrac{3}{2} \\ \left(-\tfrac{3}{2}\right)^2 &= \tfrac{9}{4} \end{aligned}$$

Add the result to both sides of the equal sign.

$$\begin{aligned} x^2 - 3x + \left(-\tfrac{3}{2}\right)^2 &= 5 + \left(-\tfrac{3}{2}\right)^2 \\ x^2 - 3x + \tfrac{9}{4} &= 5 + \tfrac{9}{4} \end{aligned}$$

Factor the left side as a perfect square and simplify the right side.

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

Use the square root property and solve.

$$\begin{aligned} \sqrt{\left(x - \tfrac{3}{2}\right)^2} &= \pm\sqrt{\tfrac{29}{4}} \\ \left(x - \tfrac{3}{2}\right) &= \pm\tfrac{\sqrt{29}}{2} \\ x &= \tfrac{3}{2} \pm \tfrac{\sqrt{29}}{2} \end{aligned}$$

The solutions are $\frac{3+\sqrt{29}}{2}$ and $\frac{3-\sqrt{29}}{2}$.

TRY IT #7 Solve by completing the square: $x^2 - 6x = 13$.

Using the Quadratic Formula

The fourth method of solving a quadratic equation is by using the quadratic formula, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by completing the square. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by -1 and obtain a positive a. Given $ax^2 + bx + c = 0$, $a \neq 0$, we will complete the square as follows:

1. First, move the constant term to the right side of the equal sign:
$$ax^2 + bx = -c$$
2. As we want the leading coefficient to equal 1, divide through by a:
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
3. Then, find $\frac{1}{2}$ of the middle term, and add $\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$ to both sides of the equal sign:
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

5. Now, use the square root property, which gives

$$\begin{aligned} x+\tfrac{b}{2a} &= \pm\sqrt{\tfrac{b^2-4ac}{4a^2}} \\ x+\tfrac{b}{2a} &= \tfrac{\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

6. Finally, add $-\frac{b}{2a}$ to both sides of the equation and combine the terms on the right side. Thus,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The Quadratic Formula

Written in standard form, $ax^2+bx+c=0$, any quadratic equation can be solved using the **quadratic formula**:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

where *a*, *b*, and *c* are real numbers and $a\neq 0$.



HOW TO

Given a quadratic equation, solve it using the quadratic formula

1. Make sure the equation is in standard form: $ax^2+bx+c=0$.
2. Make note of the values of the coefficients and constant term, a, b, and c.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

EXAMPLE 9

Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation: $x^2+5x+1=0$.

Solution

Identify the coefficients: $a=1, b=5, c=1$. Then use the quadratic formula.

$$\begin{aligned} x &= \frac{-(5)\pm\sqrt{(5)^2-4(1)(1)}}{2(1)} \\ &= \frac{-5\pm\sqrt{25-4}}{2} \\ &= \frac{-5\pm\sqrt{21}}{2} \end{aligned}$$

EXAMPLE 10

Solving a Quadratic Equation with the Quadratic Formula

Use the quadratic formula to solve $x^2+x+2=0$.

Solution

First, we identify the coefficients: $a = 1, b = 1$, and $c = 2$.

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{(1)^2 - (4) \cdot (1) \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $\frac{-1+i\sqrt{7}}{2}$ and $\frac{-1-i\sqrt{7}}{2}$

TRY IT #8 Solve the quadratic equation using the quadratic formula: $9x^2 + 3x - 2 = 0$.

The Discriminant

The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical, $b^2 - 4ac$. The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. Table 1 relates the value of the discriminant to the solutions of a quadratic equation.

Value of Discriminant	Results
$b^2 - 4ac = 0$	One rational solution (double solution)
$b^2 - 4ac > 0$, perfect square	Two rational solutions
$b^2 - 4ac > 0$, not a perfect square	Two irrational solutions
$b^2 - 4ac < 0$	Two complex solutions

Table 1

The Discriminant

For $ax^2 + bx + c = 0$, where a, b, and c are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: $b^2 - 4ac$. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

EXAMPLE 11

Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

(a) $x^2 + 4x + 4 = 0$ (b) $8x^2 + 14x + 3 = 0$ (c) $3x^2 - 5x - 2 = 0$ (d) $3x^2 - 10x + 15 = 0$

Solution

Calculate the discriminant $b^2 - 4ac$ for each equation and state the expected type of solutions.

ⓐ
$x^2 + 4x + 4 = 0$

$b^2 - 4ac = (4)^2 - 4\,(1)\,(4) = 0$. There will be one rational double solution.

ⓑ
$8x^2 + 14x + 3 = 0$

$b^2 - 4ac = (14)^2 - 4\,(8)\,(3) = 100$. As 100 is a perfect square, there will be two rational solutions.

ⓒ
$3x^2 - 5x - 2 = 0$

$b^2 - 4ac = (-5)^2 - 4\,(3)\,(-2) = 49$. As 49 is a perfect square, there will be two rational solutions.

ⓓ
$3x^2 - 10x + 15 = 0$

$b^2 - 4ac = (-10)^2 - 4\,(3)\,(15) = -80$. There will be two complex solutions.

Using the Pythagorean Theorem

One of the most famous formulas in mathematics is the **Pythagorean Theorem**. It is based on a right triangle, and states the relationship among the lengths of the sides as $a^2 + b^2 = c^2$, where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations that we learned in this section to solve for the missing side.

The Pythagorean Theorem is given as

$$a^2 + b^2 = c^2$$

where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse, as shown in Figure 4.

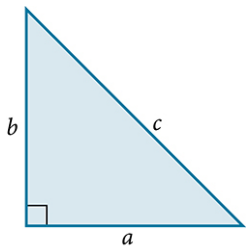


Figure 4

EXAMPLE 12

Finding the Length of the Missing Side of a Right Triangle

Find the length of the missing side of the right triangle in Figure 5.

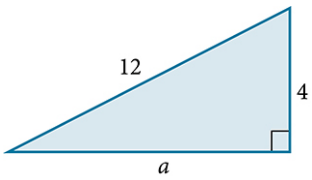


Figure 5

Solution

As we have measurements for side *b* and the hypotenuse, the missing side is *a*.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (4)^2 &= (12)^2 \\ a^2 + 16 &= 144 \\ a^2 &= 128 \\ a &= \sqrt{128} \\ &= 8\sqrt{2} \end{aligned}$$

TRY IT #9 Use the Pythagorean Theorem to solve the right triangle problem: Leg *a* measures 4 units, leg *b* measures 3 units. Find the length of the hypotenuse.

MEDIA

Access these online resources for additional instruction and practice with quadratic equations.

Solving Quadratic Equations by Factoring (https://openstax.org/l/quadreqfactor)
The Zero-Product Property (https://openstax.org/l/zeroprodprop)
Completing the Square (https://openstax.org/l/complthesqr)
Quadratic Formula with Two Rational Solutions (https://openstax.org/l/quadrformrat)
Length of a leg of a right triangle (https://openstax.org/l/leglengthtri)

2.5 SECTION EXERCISES

Verbal

1. How do we recognize when an equation is quadratic?

2. When we solve a quadratic equation, how many solutions should we always start out seeking? Explain why when solving a quadratic equation in the form $ax^2 + bx + c = 0$ we may graph the equation $y = ax^2 + bx + c$ and have no zeroes (*x*-intercepts).

3. When we solve a quadratic equation by factoring, why do we move all terms to one side, having zero on the other side?

4. In the quadratic formula, what is the name of the expression under the radical sign $b^2 - 4ac$, and how does it determine the number of and nature of our solutions?

5. Describe two scenarios where using the square root property to solve a quadratic equation would be the most efficient method.

Algebraic

For the following exercises, solve the quadratic equation by factoring.

6. $x^2 + 4x - 21 = 0$

7. $x^2 - 9x + 18 = 0$

8. $2x^2 + 9x - 5 = 0$

9. $6x^2 + 17x + 5 = 0$

10. $4x^2 - 12x + 8 = 0$

11. $3x^2 - 75 = 0$

12. $8x^2 + 6x - 9 = 0$

13. $4x^2 = 9$

14. $2x^2 + 14x = 36$

15. $5x^2 = 5x + 30$

16. $4x^2 = 5x$

17. $7x^2 + 3x = 0$

18. $\frac{x}{3} - \frac{9}{x} = 2$

For the following exercises, solve the quadratic equation by using the square root property.

19. $x^2 = 36$

20. $x^2 = 49$

21. $(x-1)^2 = 25$

22. $(x-3)^2 = 7$

23. $(2x+1)^2 = 9$

24. $(x-5)^2 = 4$

For the following exercises, solve the quadratic equation by completing the square. Show each step.

25. $x^2 - 9x - 22 = 0$

26. $2x^2 - 8x - 5 = 0$

27. $x^2 - 6x = 13$

28. $x^2 + \frac{2}{3}x - \frac{1}{3} = 0$

29. $2 + z = 6z^2$

30. $6p^2 + 7p - 20 = 0$

31. $2x^2 - 3x - 1 = 0$

For the following exercises, determine the discriminant, and then state how many solutions there are and the nature of the solutions. Do not solve.

32. $2x^2 - 6x + 7 = 0$

33. $x^2 + 4x + 7 = 0$

34. $3x^2 + 5x - 8 = 0$

35. $9x^2 - 30x + 25 = 0$

36. $2x^2 - 3x - 7 = 0$

37. $6x^2 - x - 2 = 0$

For the following exercises, solve the quadratic equation by using the quadratic formula. If the solutions are not real, state No Real Solution.

38. $2x^2 + 5x + 3 = 0$

39. $x^2 + x = 4$

40. $2x^2 - 8x - 5 = 0$

41. $3x^2 - 5x + 1 = 0$

42. $x^2 + 4x + 2 = 0$

43. $4 + \frac{1}{x} - \frac{1}{x^2} = 0$

Technology

*For the following exercises, enter the expressions into your graphing utility and find the zeroes to the equation (the x-intercepts) by using **2nd CALC 2:zero**. Recall finding zeroes will ask left bound (move your cursor to the left of the zero,enter), then right bound (move your cursor to the right of the zero,enter), then guess (move your cursor between the bounds near the zero, enter). Round your answers to the nearest thousandth.*

44. $Y_1 = 4x^2 + 3x - 2$

45. $Y_1 = -3x^2 + 8x - 1$

46. $Y_1 = 0.5x^2 + x - 7$

47. To solve the quadratic equation $x^2 + 5x - 7 = 4$, we can graph these two equations

$Y_1 = x^2 + 5x - 7$
$Y_2 = 4$

and find the points of intersection. Recall 2nd CALC 5:intersection. Do this and find the solutions to the nearest tenth.

48. To solve the quadratic equation $0.3x^2 + 2x - 4 = 2$, we can graph these two equations

$Y_1 = 0.3x^2 + 2x - 4$
$Y_2 = 2$

and find the points of intersection. Recall 2nd CALC 5:intersection. Do this and find the solutions to the nearest tenth.

Extensions

49. Beginning with the general form of a quadratic equation, $ax^2 + bx + c = 0$, solve for x by using the completing the square method, thus deriving the quadratic formula.

50. Show that the sum of the two solutions to the quadratic equation is $-\frac{b}{a}$.

51. A person has a garden that has a length 10 feet longer than the width. Set up a quadratic equation to find the dimensions of the garden if its area is 119 ft.^{2}. Solve the quadratic equation to find the length and width.

52. Abercrombie and Fitch stock had a price given as $P = 0.2t^2 - 5.6t + 50.2$, where t is the time in months from 1999 to 2001. ($t = 1$ is January 1999). Find the two months in which the price of the stock was \$30.

53. Suppose that an equation is given $p = -2x^2 + 280x - 1000$, where x represents the number of items sold at an auction and p is the profit made by the business that ran the auction. How many items sold would make this profit a maximum? Solve this by graphing the expression in your graphing utility and finding the maximum using 2nd CALC maximum. To obtain a good window for the curve, set x [0,200] and y [0,10000].

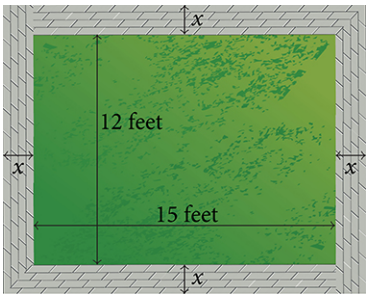
Real-World Applications

54. A formula for the normal systolic blood pressure for a man age A, measured in mmHg, is given as $P = 0.006A^2 - 0.02A + 120$. Find the age to the nearest year of a man whose normal blood pressure measures 125 mmHg.

55. The cost function for a certain company is $C = 60x + 300$ and the revenue is given by $R = 100x - 0.5x^2$. Recall that profit is revenue minus cost. Set up a quadratic equation and find two values of x (production level) that will create a profit of \$300.

56. A falling object travels a distance given by the formula $d = 5t + 16t^2$ ft, where t is measured in seconds. How long will it take for the object to travel 74 ft?

57. A vacant lot is being converted into a community garden. The garden and the walkway around its perimeter have an area of 378 ft^2. Find the width of the walkway if the garden is 12 ft. wide by 15 ft. long.



58. An epidemiological study of the spread of a certain influenza strain that hit a small school population found that the total number of students, P, who contracted the flu t days after it broke out is given by the model $P = -t^2 + 13t + 130$, where $1 \leq t \leq 6$. Find the day that 160 students had the flu. Recall that the restriction on t is at most 6.

2.6 Other Types of Equations

Learning Objectives

In this section, you will:

- Solve equations involving rational exponents.
- Solve equations using factoring.
- Solve radical equations.
- Solve absolute value equations.
- Solve other types of equations.

We have solved linear equations, rational equations, and quadratic equations using several methods. However, there are many other types of equations, and we will investigate a few more types in this section. We will look at equations involving rational exponents, polynomial equations, radical equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules. We will learn some new techniques as they apply to certain equations, but the algebra never changes.

Solving Equations Involving Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$; $8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$. The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

We can solve equations in which a variable is raised to a rational exponent by raising both sides of the equation to the reciprocal of the exponent. The reason we raise the equation to the reciprocal of the exponent is because we want to eliminate the exponent on the variable term, and a number multiplied by its reciprocal equals 1. For example, $\frac{2}{3}\left(\frac{3}{2}\right) = 1$, $3\left(\frac{1}{3}\right) = 1$, and so on.

Rational Exponents

A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

EXAMPLE 1

Evaluating a Number Raised to a Rational Exponent

Evaluate $8^{\frac{2}{3}}$.

Solution

Whether we take the root first or the power first depends on the number. It is easy to find the cube root of 8, so rewrite $8^{\frac{2}{3}}$ as $\left(8^{\frac{1}{3}}\right)^2$.

$$\begin{aligned}\left(8^{\frac{1}{3}}\right)^2 &= (2)^2\\ &= 4\end{aligned}$$

TRY IT #1 Evaluate $64^{-\frac{1}{3}}$.

EXAMPLE 2

Solve the Equation Including a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{5}{4}} = 32$.

Solution

The way to remove the exponent on x is by raising both sides of the equation to a power that is the reciprocal of $\frac{5}{4}$, which is $\frac{4}{5}$.

$$\begin{aligned}x^{\frac{5}{4}} &= 32\\ \left(x^{\frac{5}{4}}\right)^{\frac{4}{5}} &= (32)^{\frac{4}{5}}\\ x &= (2)^4 \qquad \text{The fifth root of 32 is 2.}\\ &= 16\end{aligned}$$

TRY IT #2 Solve the equation $x^{\frac{3}{2}} = 125$.

EXAMPLE 3

Solving an Equation Involving Rational Exponents and Factoring

Solve $3x^{\frac{3}{4}} = x^{\frac{1}{2}}$.

Solution

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.

$$\begin{aligned}3x^{\frac{3}{4}} - \left(x^{\frac{1}{2}}\right) &= x^{\frac{1}{2}} - \left(x^{\frac{1}{2}}\right)\\ 3x^{\frac{3}{4}} - x^{\frac{1}{2}} &= 0\end{aligned}$$

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. Rewrite $x^{\frac{1}{2}}$ as $x^{\frac{2}{4}}$. Then, factor out $x^{\frac{2}{4}}$ from both terms on the left.

$$\begin{aligned} 3x^{\frac{3}{4}} - x^{\frac{2}{4}} &= 0 \\ x^{\frac{2}{4}}\left(3x^{\frac{1}{4}} - 1\right) &= 0 \end{aligned}$$

Where did $x^{\frac{1}{4}}$ come from? Remember, when we multiply two numbers with the same base, we add the exponents. Therefore, if we multiply $x^{\frac{2}{4}}$ back in using the distributive property, we get the expression we had before the factoring, which is what should happen. We need an exponent such that when added to $\frac{2}{4}$ equals $\frac{3}{4}$. Thus, the exponent on x in the parentheses is $\frac{1}{4}$.

Let us continue. Now we have two factors and can use the zero factor theorem.

$$\begin{aligned} x^{\frac{2}{4}}\left(3x^{\frac{1}{4}} - 1\right) &= 0 \\ x^{\frac{2}{4}} &= 0 \\ x &= 0 \\ 3x^{\frac{1}{4}} - 1 &= 0 \\ 3x^{\frac{1}{4}} &= 1 \\ x^{\frac{1}{4}} &= \tfrac{1}{3} && \text{Divide both sides by 3.} \\ \left(x^{\frac{1}{4}}\right)^4 &= \left(\tfrac{1}{3}\right)^4 && \text{Raise both sides to the reciprocal of } \tfrac{1}{4}. \\ x &= \tfrac{1}{81} \end{aligned}$$

The two solutions are 0 and $\frac{1}{81}$.

TRY IT #3 Solve: $(x+5)^{\frac{3}{2}} = 8$.

Solving Equations Using Factoring

We have used factoring to solve quadratic equations, but it is a technique that we can use with many types of polynomial equations, which are equations that contain a string of terms including numerical coefficients and variables. When we are faced with an equation containing polynomials of degree higher than 2, we can often solve them by factoring.

Polynomial Equations

A polynomial of degree n is an expression of the type

$$a_n x^n + a_{n-1} x^{n-1} + \quad \cdot \quad \cdot \quad \cdot \quad + a_2 x^2 + a_1 x + a_0$$

where n is a positive integer and $a_n, \ldots, a_0$ are real numbers and $a_n \neq 0$.

Setting the polynomial equal to zero gives a **polynomial equation**. The total number of solutions (real and complex) to a polynomial equation is equal to the highest exponent n.

EXAMPLE 4

Solving a Polynomial by Factoring

Solve the polynomial by factoring: $5x^4 = 80x^2$.

Solution

First, set the equation equal to zero. Then factor out what is common to both terms, the GCF.

$$\begin{aligned} 5x^4 - 80x^2 &= 0 \\ 5x^2\left(x^2 - 16\right) &= 0 \end{aligned}$$

Notice that we have the difference of squares in the factor $x^2 - 16$, which we will continue to factor and obtain two solutions. The first term, $5x^2$, generates, technically, two solutions as the exponent is 2, but they are the same solution.

$$\begin{aligned} 5x^2 &= 0 \\ x &= 0 \\ x^2 - 16 &= 0 \\ (x-4)(x+4) &= 0 \\ x &= 4 \\ x &= -4 \end{aligned}$$

The solutions are 0 (double solution), 4, and -4.

Analysis

We can see the solutions on the graph in Figure 1. The *x*-coordinates of the points where the graph crosses the *x*-axis are the solutions—the *x*-intercepts. Notice on the graph that at the solution 0, the graph touches the *x*-axis and bounces back. It does not cross the *x*-axis. This is typical of double solutions.

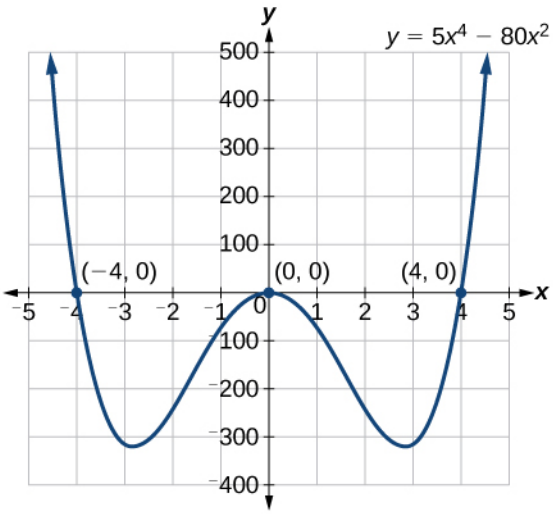


Figure 1

TRY IT #4 Solve by factoring: $12x^4 = 3x^2$.

EXAMPLE 5

Solve a Polynomial by Grouping

Solve a polynomial by grouping: $x^3 + x^2 - 9x - 9 = 0$.

Solution

This polynomial consists of 4 terms, which we can solve by grouping. Grouping procedures require factoring the first two terms and then factoring the last two terms. If the factors in the parentheses are identical, we can continue the process and solve, unless more factoring is suggested.

$$\begin{aligned} x^3 + x^2 - 9x - 9 &= 0 \\ x^2(x+1) - 9(x+1) &= 0 \\ \left(x^2 - 9\right)(x+1) &= 0 \end{aligned}$$

The grouping process ends here, as we can factor $x^2 - 9$ using the difference of squares formula.

$$\begin{aligned}(x^2-9)(x+1) &= 0\\ (x-3)(x+3)(x+1) &= 0\\ x &= 3\\ x &= -3\\ x &= -1\end{aligned}$$

The solutions are 3, -3, and -1. Note that the highest exponent is 3 and we obtained 3 solutions. We can see the solutions, the *x*-intercepts, on the graph in Figure 2.

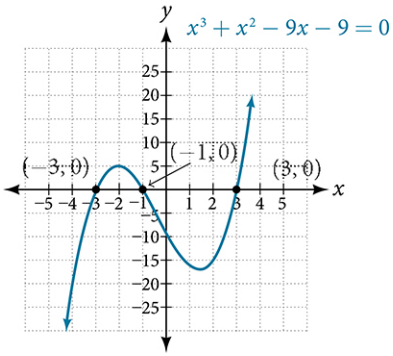


Figure 2

Analysis

We looked at solving quadratic equations by factoring when the leading coefficient is 1. When the leading coefficient is not 1, we solved by grouping. Grouping requires four terms, which we obtained by splitting the linear term of quadratic equations. We can also use grouping for some polynomials of degree higher than 2, as we saw here, since there were already four terms.

Solving Radical Equations

Radical equations are equations that contain variables in the radicand (the expression under a radical symbol), such as

$$\begin{aligned}\sqrt{3x+18} &= x\\ \sqrt{x+3} &= x-3\\ \sqrt{x+5}-\sqrt{x-3} &= 2\end{aligned}$$

Radical equations may have one or more radical terms, and are solved by eliminating each radical, one at a time. We have to be careful when solving radical equations, as it is not unusual to find **extraneous solutions**, roots that are not, in fact, solutions to the equation. These solutions are not due to a mistake in the solving method, but result from the process of raising both sides of an equation to a power. However, checking each answer in the original equation will confirm the true solutions.

Radical Equations

An equation containing terms with a variable in the radicand is called a **radical equation**.

HOW TO

Given a radical equation, solve it.

1. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
2. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an *n*th root radical, raise both sides to the *n*th power. Doing so eliminates the radical symbol.

3. Solve the remaining equation.
4. If a radical term still remains, repeat steps 1–2.
5. Confirm solutions by substituting them into the original equation.

EXAMPLE 6

Solving an Equation with One Radical

Solve $\sqrt{15-2x}=x$.

Solution

The radical is already isolated on the left side of the equal side, so proceed to square both sides.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \left(\sqrt{15-2x}\right)^2 &= (x)^2\\ 15-2x &= x^2\end{aligned}$$

We see that the remaining equation is a quadratic. Set it equal to zero and solve.

$$\begin{aligned}0 &= x^2+2x-15\\ &= (x+5)(x-3)\\ x &= -5\\ x &= 3\end{aligned}$$

The proposed solutions are -5 and 3. Let us check each solution back in the original equation. First, check $x=-5$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(-5)} &= -5\\ \sqrt{25} &= -5\\ 5 &\neq -5\end{aligned}$$

This is an extraneous solution. While no mistake was made solving the equation, we found a solution that does not satisfy the original equation.

Check $x=3$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(3)} &= 3\\ \sqrt{9} &= 3\\ 3 &= 3\end{aligned}$$

The solution is 3.

TRY IT #5 Solve the radical equation: $\sqrt{x+3}=3x-1$

EXAMPLE 7

Solving a Radical Equation Containing Two Radicals

Solve $\sqrt{2x+3}+\sqrt{x-2}=4$.

Solution

As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4-\sqrt{x-2} && \text{Subtract } \sqrt{x-2} \text{ from both sides.} \\ \left(\sqrt{2x+3}\right)^2 &= \left(4-\sqrt{x-2}\right)^2 && \text{Square both sides.}\end{aligned}$$

Use the perfect square formula to expand the right side: $(a-b)^2 = a^2-2ab+b^2$.

$$\begin{aligned}2x+3 &= (4)^2-2(4)\sqrt{x-2}+\left(\sqrt{x-2}\right)^2 \\ 2x+3 &= 16-8\sqrt{x-2}+(x-2) \\ 2x+3 &= 14+x-8\sqrt{x-2} && \text{Combine like terms.} \\ x-11 &= -8\sqrt{x-2} && \text{Isolate the second radical.} \\ (x-11)^2 &= \left(-8\sqrt{x-2}\right)^2 && \text{Square both sides.} \\ x^2-22x+121 &= 64(x-2)\end{aligned}$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.

$$\begin{aligned}x^2-22x+121 &= 64x-128 \\ x^2-86x+249 &= 0 \\ (x-3)(x-83) &= 0 && \text{Factor and solve.} \\ x &= 3 \\ x &= 83\end{aligned}$$

The proposed solutions are 3 and 83. Check each solution in the original equation.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4-\sqrt{x-2} \\ \sqrt{2(3)+3} &= 4-\sqrt{(3)-2} \\ \sqrt{9} &= 4-\sqrt{1} \\ 3 &= 3\end{aligned}$$

One solution is 3.

Check $x = 83$.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4-\sqrt{x-2} \\ \sqrt{2(83)+3} &= 4-\sqrt{(83-2)} \\ \sqrt{169} &= 4-\sqrt{81} \\ 13 &\neq -5\end{aligned}$$

The only solution is 3. We see that 83 is an extraneous solution.

TRY IT #6 Solve the equation with two radicals: $\sqrt{3x+7}+\sqrt{x+2}=1$.

Solving an Absolute Value Equation

Next, we will learn how to solve an absolute value equation. To solve an equation such as $|2x-6|=8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or −8. This leads to two different equations we can solve independently.

$$\begin{aligned}2x-6 &= 8 \\ 2x &= 14 \\ x &= 7\end{aligned} \qquad \text{or} \qquad \begin{aligned}2x-6 &= -8 \\ 2x &= -2 \\ x &= -1\end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

Absolute Value Equations

The absolute value of x is written as $|x|$. It has the following properties:

$$\text{If } x \geq 0, \text{ then } |x| = x.$$
$$\text{If } x < 0, \text{ then } |x| = -x.$$

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

An **absolute value equation** in the form $|ax + b| = c$ has the following properties:

$$\text{If } c < 0, |ax + b| = c \text{ has no solution.}$$
$$\text{If } c = 0, |ax + b| = c \text{ has one solution.}$$
$$\text{If } c > 0, |ax + b| = c \text{ has two solutions.}$$



HOW TO

Given an absolute value equation, solve it.

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c > 0$, write and solve two equations: $ax + b = c$ and $ax + b = -c$.

EXAMPLE 8

Solving Absolute Value Equations

Solve the following absolute value equations:

ⓐ $|6x + 4| = 8$ ⓑ $|3x + 4| = -9$ ⓒ $|3x - 5| - 4 = 6$ ⓓ $|-5x + 10| = 0$

Solution

ⓐ

$|6x + 4| = 8$

Write two equations and solve each:

$$\begin{aligned} 6x + 4 &= 8 \\ 6x &= 4 \\ x &= \tfrac{2}{3} \end{aligned} \qquad \begin{aligned} 6x + 4 &= -8 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

The two solutions are $\frac{2}{3}$ and -2.

ⓑ

$|3x + 4| = -9$

There is no solution as an absolute value cannot be negative.

ⓒ

$|3x - 5| - 4 = 6$

Isolate the absolute value expression and then write two equations.

$$\begin{aligned} |3x - 5| - 4 &= 6 \\ |3x - 5| &= 10 \end{aligned}$$

$$\begin{aligned} 3x - 5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned} \qquad \begin{aligned} 3x - 5 &= -10 \\ 3x &= -5 \\ x &= -\tfrac{5}{3} \end{aligned}$$

There are two solutions: 5, and $-\frac{5}{3}$.

ⓓ
(d) $|-5x+10|=0$

The equation is set equal to zero, so we have to write only one equation.

$$\begin{aligned} -5x+10 &= 0 \\ -5x &= -10 \\ x &= 2 \end{aligned}$$

There is one solution: 2.

TRY IT #7 Solve the absolute value equation: $|1-4x|+8=13$.

Solving Other Types of Equations

There are many other types of equations in addition to the ones we have discussed so far. We will see more of them throughout the text. Here, we will discuss equations that are in quadratic form, and rational equations that result in a quadratic.

Solving Equations in Quadratic Form

Equations in quadratic form are equations with three terms. The first term has a power other than 2. The middle term has an exponent that is one-half the exponent of the leading term. The third term is a constant. We can solve equations in this form as if they were quadratic. A few examples of these equations include $x^4-5x^2+4=0$, $x^6+7x^3-8=0$, and $x^{\frac{2}{3}}+4x^{\frac{1}{3}}+2=0$. In each one, doubling the exponent of the middle term equals the exponent on the leading term. We can solve these equations by substituting a variable for the middle term.

Quadratic Form

If the exponent on the middle term is one-half of the exponent on the leading term, we have an **equation in quadratic form**, which we can solve as if it were a quadratic. We substitute a variable for the middle term to solve equations in quadratic form.

HOW TO

Given an equation quadratic in form, solve it.

1. Identify the exponent on the leading term and determine whether it is double the exponent on the middle term.
2. If it is, substitute a variable, such as u, for the variable portion of the middle term.
3. Rewrite the equation so that it takes on the standard form of a quadratic.
4. Solve using one of the usual methods for solving a quadratic.
5. Replace the substitution variable with the original term.
6. Solve the remaining equation.

EXAMPLE 9

Solving a Fourth-degree Equation in Quadratic Form

Solve this fourth-degree equation: $3x^4-2x^2-1=0$.

Solution

This equation fits the main criteria, that the power on the leading term is double the power on the middle term. Next, we will make a substitution for the variable term in the middle. Let $u=x^2$. Rewrite the equation in u.

$$3u^2-2u-1=0$$

Now solve the quadratic.

$$\begin{aligned} 3u^2 - 2u - 1 &= 0 \\ (3u+1)(u-1) &= 0 \end{aligned}$$

Solve each factor and replace the original term for u.

$$\begin{aligned} 3u + 1 &= 0 \\ 3u &= -1 \\ u &= -\tfrac{1}{3} \\ x^2 &= -\tfrac{1}{3} \\ x &= \pm i\sqrt{\tfrac{1}{3}} \end{aligned}$$

$$\begin{aligned} u - 1 &= 0 \\ u &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

The solutions are $\pm i\sqrt{\frac{1}{3}}$ and ± 1.

TRY IT #8 Solve using substitution: $x^4 - 8x^2 - 9 = 0$.

EXAMPLE 10

Solving an Equation in Quadratic Form Containing a Binomial

Solve the equation in quadratic form: $(x+2)^2 + 11(x+2) - 12 = 0$.

Solution

This equation contains a binomial in place of the single variable. The tendency is to expand what is presented. However, recognizing that it fits the criteria for being in quadratic form makes all the difference in the solving process. First, make a substitution, letting $u = x + 2$. Then rewrite the equation in u.

$$\begin{aligned} u^2 + 11u - 12 &= 0 \\ (u+12)(u-1) &= 0 \end{aligned}$$

Solve using the zero-factor property and then replace u with the original expression.

$$\begin{aligned} u + 12 &= 0 \\ u &= -12 \\ x + 2 &= -12 \\ x &= -14 \end{aligned}$$

The second factor results in

$$\begin{aligned} u - 1 &= 0 \\ u &= 1 \\ x + 2 &= 1 \\ x &= -1 \end{aligned}$$

We have two solutions: -14, and -1.

TRY IT #9 Solve: $(x-5)^2 - 4(x-5) - 21 = 0$.

Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When this happens,

we continue the solution by simplifying the quadratic equation by one of the methods we have seen. It may turn out that there is no solution.

EXAMPLE 11

Solving a Rational Equation Leading to a Quadratic

Solve the following rational equation: $\frac{-4x}{x-1}+\frac{4}{x+1}=\frac{-8}{x^2-1}$.

Solution

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^2-1=(x+1)(x-1)$. Then, the LCD is $(x+1)(x-1)$. Next, we multiply the whole equation by the LCD.

$$\begin{aligned}(x+1)(x-1)\left[\frac{-4x}{x-1}+\frac{4}{x+1}\right] &= \left[\frac{-8}{(x+1)(x-1)}\right](x+1)(x-1)\\ -4x(x+1)+4(x-1) &= -8\\ -4x^2-4x+4x-4 &= -8\\ -4x^2+4 &= 0\\ -4\left(x^2-1\right) &= 0\\ -4(x+1)(x-1) &= 0\\ x &= -1\\ x &= 1\end{aligned}$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

TRY IT #10 Solve $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$.

MEDIA

Access these online resources for additional instruction and practice with different types of equations.

Rational Equation with no Solution (https://openstax.org/l/rateqnosoln)
Solving equations with rational exponents using reciprocal powers (https://openstax.org/l/ratexprecpexp)
Solving radical equations part 1 of 2 (https://openstax.org/l/radeqsolvepart1)
Solving radical equations part 2 of 2 (https://openstax.org/l/radeqsolvepart2)

2.6 SECTION EXERCISES

Verbal

1. In a radical equation, what does it mean if a number is an extraneous solution?

2. Explain why possible solutions *must* be checked in radical equations.

3. Your friend tries to calculate the value $-9^{\frac{3}{2}}$ and keeps getting an ERROR message. What mistake are they probably making?

4. Explain why $|2x+5|=-7$ has no solutions.

5. Explain how to change a rational exponent into the correct radical expression.

Algebraic

For the following exercises, solve the rational exponent equation. Use factoring where necessary.

6. $x^{\frac{2}{3}}=16$

7. $x^{\frac{3}{4}}=27$

8. $2x^{\frac{1}{2}}-x^{\frac{1}{4}}=0$

9. $(x-1)^{\frac{3}{4}}=8$ **10.** $(x+1)^{\frac{2}{3}}=4$ **11.** $x^{\frac{2}{3}}-5x^{\frac{1}{3}}+6=0$

12. $x^{\frac{7}{3}}-3x^{\frac{4}{3}}-4x^{\frac{1}{3}}=0$

For the following exercises, solve the following polynomial equations by grouping and factoring.

13. $x^3+2x^2-x-2=0$ **14.** $3x^3-6x^2-27x+54=0$ **15.** $4y^3-9y=0$

16. $x^3+3x^2-25x-75=0$ **17.** $m^3+m^2-m-1=0$ **18.** $2x^5-14x^3=0$

19. $5x^3+45x=2x^2+18$

For the following exercises, solve the radical equation. Be sure to check all solutions to eliminate extraneous solutions.

20. $\sqrt{3x-1}-2=0$ **21.** $\sqrt{x-7}=5$ **22.** $\sqrt{x-1}=x-7$

23. $\sqrt{3t+5}=7$ **24.** $\sqrt{t+1}+9=7$ **25.** $\sqrt{12-x}=x$

26. $\sqrt{2x+3}-\sqrt{x+2}=2$ **27.** $\sqrt{3x+7}+\sqrt{x+2}=1$ **28.** $\sqrt{2x+3}-\sqrt{x+1}=1$

For the following exercises, solve the equation involving absolute value.

29. $|3x-4|=8$ **30.** $|2x-3|=-2$ **31.** $|1-4x|-1=5$

32. $|4x+1|-3=6$ **33.** $|2x-1|-7=-2$ **34.** $|2x+1|-2=-3$

35. $|x+5|=0$ **36.** $-|2x+1|=-3$

For the following exercises, solve the equation by identifying the quadratic form. Use a substitute variable and find all real solutions by factoring.

37. $x^4-10x^2+9=0$ **38.** $4(t-1)^2-9(t-1)=-2$ **39.** $\left(x^2-1\right)^2+\left(x^2-1\right)-12=0$

40. $(x+1)^2-8(x+1)-9=0$ **41.** $(x-3)^2-4=0$

Extensions

For the following exercises, solve for the unknown variable.

42. $x^{-2}-x^{-1}-12=0$ **43.** $\sqrt{|x|^2}=x$ **44.** $t^{10}-2t^5+1=0$

45. $\left|x^2+2x-36\right|=12$

Real-World Applications

For the following exercises, use the model for the period of a pendulum, T, such that $T=2\pi\sqrt{\frac{L}{g}}$, where the length of the pendulum is L and the acceleration due to gravity is g.

46. If the acceleration due to gravity is 9.8 m/s^2 and the period equals 1 s, find the length to the nearest cm (100 cm = 1 m).

47. If the gravity is 32 ft/s^2 and the period equals 1 s, find the length to the nearest in. (12 in. = 1 ft). Round your answer to the nearest in.

For the following exercises, use a model for body surface area, BSA, such that $BSA=\sqrt{\frac{wh}{3600}}$, where w = weight in kg and h = height in cm.

48. Find the height of a 72-kg female to the nearest cm whose $BSA=1.8$.

49. Find the weight of a 177-cm male to the nearest kg whose $BSA=2.1$.

2.7 Linear Inequalities and Absolute Value Inequalities

Learning Objectives

In this section, you will:

- Use interval notation
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.



Figure 1

It is not easy to make the honor roll at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in Figure 2**.** The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

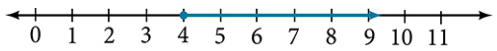


Figure 2

We can use set-builder notation: $\{x|x \geq 4\}$, which translates to "all real numbers *x* such that *x* is greater than or equal to 4." Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $\left[4, \infty\right)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1, 0)$, all real numbers between, but not including -1 and 0; and $\left(-\infty, 1\right]$, all real numbers less than and including 1. Table 1 outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between *a* and *b*, but not including *a* or *b*	$\{x\|a < x < b\}$	(a, b)
All real numbers greater than *a*, but not including *a*	$\{x\|x > a\}$	$\left(a, \infty\right)$
All real numbers less than *b*, but not including *b*	$\{x\|x < b\}$	$\left(-\infty, b\right)$

Table 1

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers greater than a, including a	$\{x \mid x \geq a\}$	$\left[a, \infty\right)$
All real numbers less than b, including b	$\{x \mid x \leq b\}$	$\left(-\infty, b\right]$
All real numbers between a and b, including a	$\{x \mid a \leq x < b\}$	$[a, b)$
All real numbers between a and b, including b	$\{x \mid a < x \leq b\}$	$(a, b]$
All real numbers between a and b, including a and b	$\{x \mid a \leq x \leq b\}$	$[a, b]$
All real numbers less than a or greater than b	$\{x \mid x < a \text{ or } x > b\}$	$\left(-\infty, a\right) \cup \left(b, \infty\right)$
All real numbers	$\{x \mid x \text{ is all real numbers}\}$	$\left(-\infty, \infty\right)$

Table 1

EXAMPLE 1

Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a*

Use interval notation to indicate all real numbers greater than or equal to -2.

Solution

Use a bracket on the left of -2 and parentheses after infinity: $\left[-2, \infty\right)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

TRY IT #1 Use interval notation to indicate all real numbers between and including -3 and 5.

EXAMPLE 2

Using Interval Notation to Express All Real Numbers Less Than or Equal to *a* or Greater Than or Equal to *b*

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

Solution

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1, which is written as $\left(-\infty, -1\right]$.

The second interval must show all real numbers greater than or equal to 1, which is written as $\left[1, \infty\right)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, $\cup$, between the two intervals.

$$\left(-\infty, -1\right] \cup \left[1, \infty\right)$$

TRY IT #2 Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

Properties of Inequalities

Addition Property	If $a < b$, then $a + c < b + c$.
Multiplication Property	If $a < b$ and $c > 0$, then $ac < bc$. If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

EXAMPLE 3

Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

ⓐ $x - 15 < 4$ ⓑ $6 \geq x - 1$ ⓒ $x + 7 > 9$

Solution

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

ⓐ

$$\begin{aligned} x - 15 &< 4 \\ x - 15 + 15 &< 4 + 15 \quad \text{Add 15 to both sides.} \\ x &< 19 \end{aligned}$$

ⓑ

$$\begin{aligned} 6 &\geq x - 1 \\ 6 + 1 &\geq x - 1 + 1 \quad \text{Add 1 to both sides.} \\ 7 &\geq x \end{aligned}$$

ⓒ

$$\begin{aligned} x + 7 &> 9 \\ x + 7 - 7 &> 9 - 7 \quad \text{Subtract 7 from both sides.} \\ x &> 2 \end{aligned}$$

TRY IT #3 Solve: $3x - 2 < 1$.

EXAMPLE 4

Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

ⓐ $3x < 6$ ⓑ $-2x - 1 \geq 5$ ⓒ $5 - x > 10$

Solution

ⓐ

$$\begin{aligned} 3x &< 6 \\ \tfrac{1}{3}(3x) &< (6)\tfrac{1}{3} \\ x &< 2 \end{aligned}$$

ⓑ

$$\begin{aligned} -2x - 1 &\geq 5 \\ -2x &\geq 6 \\ \left(-\tfrac{1}{2}\right)(-2x) &\geq (6)\left(-\tfrac{1}{2}\right) \quad \text{Multiply by } -\tfrac{1}{2}. \\ x &\leq -3 \quad \text{Reverse the inequality.} \end{aligned}$$

ⓒ

$$\begin{aligned} 5 - x &> 10 \\ -x &> 5 \\ (-1)(-x) &> (5)(-1) \quad \text{Multiply by } -1. \\ x &< -5 \quad \text{Reverse the inequality.} \end{aligned}$$

TRY IT #4 Solve: $4x + 7 \geq 2x - 3$.

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with

equations; we combine like terms and perform operations. To solve, we isolate the variable.

EXAMPLE 5

Solving an Inequality Algebraically

Solve the inequality: $13 - 7x \geq 10x - 4$.

Solution

Solving this inequality is similar to solving an equation up until the last step.

$$\begin{aligned} 13 - 7x &\geq 10x - 4 \\ 13 - 17x &\geq -4 && \text{Move variable terms to one side of the inequality.} \\ -17x &\geq -17 && \text{Isolate the variable term.} \\ x &\leq 1 && \text{Dividing both sides by} -17 \text{ reverses the inequality.} \end{aligned}$$

The solution set is given by the interval $\left(-\infty, 1\right]$, or all real numbers less than and including 1.

TRY IT #5 Solve the inequality and write the answer using interval notation: $-x + 4 < \frac{1}{2}x + 1$.

EXAMPLE 6

Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.

Solution

We begin solving in the same way we do when solving an equation.

$$\begin{aligned} -\tfrac{3}{4}x &\geq -\tfrac{5}{8} + \tfrac{2}{3}x \\ -\tfrac{3}{4}x - \tfrac{2}{3}x &\geq -\tfrac{5}{8} && \text{Put variable terms on one side.} \\ -\tfrac{9}{12}x - \tfrac{8}{12}x &\geq -\tfrac{5}{8} && \text{Write fractions with common denominator.} \\ -\tfrac{17}{12}x &\geq -\tfrac{5}{8} \\ x &\leq -\tfrac{5}{8}\left(-\tfrac{12}{17}\right) && \text{Multiplying by a negative number reverses the inequality.} \\ x &\leq \tfrac{15}{34} \end{aligned}$$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

TRY IT #6 Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \leq 6$ means $4 < x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

EXAMPLE 7

Solving a Compound Inequality

Solve the compound inequality: $3 \leq 2x + 2 < 6$.

Solution

The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$$\begin{aligned} 3 &\le 2x+2 \\ 1 &\le 2x \\ \tfrac{1}{2} &\le x \end{aligned} \quad \text{and} \quad \begin{aligned} 2x+2 &< 6 \\ 2x &< 4 \\ x &< 2 \end{aligned}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \le x < 2$$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$$\begin{aligned} &3 \le 2x+2 < 6 \\ &1 \le 2x < 4 && \text{Isolate the variable term, and subtract 2 from all three parts.} \\ &\tfrac{1}{2} \le x < 2 && \text{Divide through all three parts by 2.} \end{aligned}$$

We get the same solution: $\left[\frac{1}{2}, 2\right)$.

TRY IT #7 Solve the compound inequality: $4 < 2x - 8 \le 10$.

EXAMPLE 8

Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

Solution

Let's try the first method. Write two inequalities:

$$\begin{aligned} 3 + x &> 7x - 2 \\ 3 &> 6x - 2 \\ 5 &> 6x \\ \tfrac{5}{6} &> x \\ x &< \tfrac{5}{6} \end{aligned} \quad \text{and} \quad \begin{aligned} 7x - 2 &> 5x - 10 \\ 2x - 2 &> -10 \\ 2x &> -8 \\ x &> -4 \\ -4 &< x \end{aligned}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See Figure 3.



Figure 3

TRY IT #8 Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x, as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A| < B, \quad |A| \le B, \quad |A| > B, \quad \text{or} \quad |A| \ge B,$$

Where *A*, and sometimes *B*, represents an algebraic expression dependent on a variable *x*. Solving the inequality means finding the set of all x -values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of *x*-values such that the distance between x and 600 is less than or equal to 200. We represent the distance between x and 600 as $|x-600|$, and therefore, $|x-600| \le 200$ or

$$\begin{gathered} -200 \le x-600 \le 200 \\ -200+600 \le x-600+600 \le 200+600 \\ 400 \le x \le 800 \end{gathered}$$

This means our returns would be between $400 and $800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

Absolute Value Inequalities

For an algebraic expression *X*, and $k > 0$, an **absolute value inequality** is an inequality of the form

$$\begin{aligned} |X| < k \quad &\text{is equivalent to } -k < X < k \\ |X| > k \quad &\text{is equivalent to } X < -k \quad \text{or } X > k \end{aligned}$$

These statements also apply to $|X| \le k$ and $|X| \ge k$.

EXAMPLE 9

Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in Figure 4, to represent the condition to be satisfied.

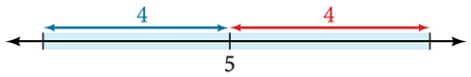


Figure 4

The distance from x to 5 can be represented using an absolute value symbol, $|x-5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x-5| \le 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$\begin{aligned} x-5 &\le 4 \quad \text{and} \quad x-5 \ge -4 \\ x &\le 9 \qquad\qquad x \ge 1 \end{aligned}$$

If the solution set is $x \le 9$ and $x \ge 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x-5| \le 4$ is equivalent to $[1, 9]$ in interval notation.

TRY IT #9 Describe all *x*-values within a distance of 3 from the number 2.

EXAMPLE 10

Solving an Absolute Value Inequality

Solve $|x-1| \le 3$.

Solution

$$|x-1| \le 3$$

$$-3 \le x-1 \le 3$$

$$-2 \le x \le 4$$

$$[-2, 4]$$

EXAMPLE 11

Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y = -\frac{1}{2}|4x-5|+3$, determine the *x*-values for which the *y*-values are negative.

Solution

We are trying to determine where $y < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$-\frac{1}{2}|4x-5| < -3 \qquad \text{Multiply both sides by –2, and reverse the inequality.}$$

$$|4x-5| > 6$$

Next, we solve for the equality $|4x-5| = 6$.

$$4x-5=6 \quad \text{or} \quad 4x-5=-6$$

$$4x=11 \quad \text{or} \quad 4x=-1$$

$$x=\frac{11}{4} \quad \text{or} \quad x=-\frac{1}{4}$$

Now, we can examine the graph to observe where the *y*-values are negative. We observe where the branches are below the *x*-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x=-\frac{1}{4}$ and $x=\frac{11}{4}$, and that the graph opens downward. See Figure 5.

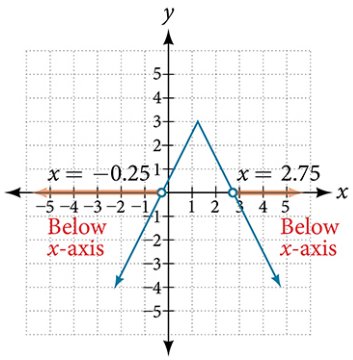


Figure 5

TRY IT #10 Solve $-2|k-4| \le -6$.

MEDIA

Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

Interval notation (https://openstax.org/l/intervalnotn)
How to solve linear inequalities (https://openstax.org/l/solvelinineq)
How to solve an inequality (https://openstax.org/l/solveineq)
Absolute value equations (https://openstax.org/l/absvaleq)
Compound inequalities (https://openstax.org/l/compndineqs)
Absolute value inequalities (https://openstax.org/l/absvalineqs)

2.7 SECTION EXERCISES

Verbal

1. When solving an inequality, explain what happened from Step 1 to Step 2:
Step 1 $-2x > 6$
Step 2 $x < -3$

2. When solving an inequality, we arrive at:
$x + 2 < x + 3$
$2 < 3$
Explain what our solution set is.

3. When writing our solution in interval notation, how do we represent all the real numbers?

4. When solving an inequality, we arrive at:
$x + 2 > x + 3$
$2 > 3$
Explain what our solution set is.

5. Describe how to graph $y = |x - 3|$

Algebraic

For the following exercises, solve the inequality. Write your final answer in interval notation.

6. $4x - 7 \le 9$

7. $3x + 2 \ge 7x - 1$

8. $-2x + 3 > x - 5$

9. $4(x + 3) \ge 2x - 1$

10. $-\frac{1}{2}x \le -\frac{5}{4} + \frac{2}{5}x$

11. $-5(x - 1) + 3 > 3x - 4 - 4x$

12. $-3(2x + 1) > -2(x + 4)$

13. $\frac{x+3}{8} - \frac{x+5}{5} \ge \frac{3}{10}$

14. $\frac{x-1}{3} + \frac{x+2}{5} \le \frac{3}{5}$

For the following exercises, solve the inequality involving absolute value. Write your final answer in interval notation.

15. $|x + 9| \ge -6$

16. $|2x + 3| < 7$

17. $|3x - 1| > 11$

18. $|2x + 1| + 1 \le 6$

19. $|x - 2| + 4 \ge 10$

20. $|-2x + 7| \le 13$

21. $|x - 7| < -4$

22. $|x - 20| > -1$

23. $\left|\frac{x-3}{4}\right| < 2$

For the following exercises, describe all the x-values within or including a distance of the given values.

24. Distance of 5 units from the number 7

25. Distance of 3 units from the number 9

26. Distance of 10 units from the number 4

27. Distance of 11 units from the number 1

For the following exercises, solve the compound inequality. Express your answer using inequality signs, and then write your answer using interval notation.

28. $-4 < 3x + 2 \le 18$

29. $3x + 1 > 2x - 5 > x - 7$

30. $3y < 5 - 2y < 7 + y$

31. $2x - 5 < -11$ or $5x + 1 \ge 6$

32. $x + 7 < x + 2$

Graphical

For the following exercises, graph the function. Observe the points of intersection and shade the x-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

33. $|x-1|>2$

34. $|x+3|\geq 5$

35. $|x+7|\leq 4$

36. $|x-2|<7$

37. $|x-2|<0$

For the following exercises, graph both straight lines (left-hand side being y1 and right-hand side being y2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the y-values of the lines.

38. $x+3<3x-4$

39. $x-2>2x+1$

40. $x+1>x+4$

41. $\frac{1}{2}x+1>\frac{1}{2}x-5$

42. $4x+1<\frac{1}{2}x+3$

Numeric

For the following exercises, write the set in interval notation.

43. $\{x|-1<x<3\}$

44. $\{x|x\geq 7\}$

45. $\{x|x<4\}$

46. $\{\ x|\ \ x\ \ \text{is all real numbers}\}$

For the following exercises, write the interval in set-builder notation.

47. $(-\infty,6)$

48. $(4,\infty)$

49. $[-3,5)$

50. $[-4,1]\cup[9,\infty)$

For the following exercises, write the set of numbers represented on the number line in interval notation.

51.



52.



53.



Technology

For the following exercises, input the left-hand side of the inequality as a Y1 graph in your graphing utility. Enter y2 = the right-hand side. Entering the absolute value of an expression is found in the MATH menu, Num, 1:abs(. Find the points of intersection, recall (2^{nd} CALC 5:intersection, 1^{st} curve, enter, 2^{nd} curve, enter, guess, enter). Copy a sketch of the graph and shade the x-axis for your solution set to the inequality. Write final answers in interval notation.

54. $|x+2|-5<2$

55. $\frac{-1}{2}|x+2|<4$

56. $|4x+1|-3>2$

57. $|x-4|<3$

58. $|x+2|\geq 5$

Extensions

59. Solve $|3x+1|=|2x+3|$

60. Solve $x^2-x>12$

61. $\frac{x-5}{x+7}\leq 0,\ x\neq -7$

62. $p=-x^2+130x-3000$ is a profit formula for a small business. Find the set of *x*-values that will keep this profit positive.

Real-World Applications

63. In chemistry the volume for a certain gas is given by $V = 20T$, where V is measured in cc and T is temperature in °C. If the temperature varies between 80°C and 120°C, find the set of volume values.

64. A basic cellular package costs $20/mo. for 60 min of calling, with an additional charge of $.30/min beyond that time.. The cost formula would be $C = 20 + .30(x - 60)$. If you have to keep your bill no greater than $50, what is the maximum calling minutes you can use?

Chapter Review

Key Terms

absolute value equation an equation in which the variable appears in absolute value bars, typically with two solutions, one accounting for the positive expression and one for the negative expression

area in square units, the area formula used in this section is used to find the area of any two-dimensional rectangular region: $A = LW$

Cartesian coordinate system a grid system designed with perpendicular axes invented by René Descartes

completing the square a process for solving quadratic equations in which terms are added to or subtracted from both sides of the equation in order to make one side a perfect square

complex conjugate a complex number containing the same terms as another complex number, but with the opposite operator. Multiplying a complex number by its conjugate yields a real number.

complex number the sum of a real number and an imaginary number; the standard form is $a + bi$, where a is the real part and b is the complex part.

complex plane the coordinate plane in which the horizontal axis represents the real component of a complex number, and the vertical axis represents the imaginary component, labeled i.

compound inequality a problem or a statement that includes two inequalities

conditional equation an equation that is true for some values of the variable

discriminant the expression under the radical in the quadratic formula that indicates the nature of the solutions, real or complex, rational or irrational, single or double roots.

distance formula a formula that can be used to find the length of a line segment if the endpoints are known

equation in two variables a mathematical statement, typically written in x and y, in which two expressions are equal

equations in quadratic form equations with a power other than 2 but with a middle term with an exponent that is one-half the exponent of the leading term

extraneous solutions any solutions obtained that are not valid in the original equation

graph in two variables the graph of an equation in two variables, which is always shown in two variables in the two-dimensional plane

identity equation an equation that is true for all values of the variable

imaginary number the square root of $-1 : i = \sqrt{-1}$.

inconsistent equation an equation producing a false result

intercepts the points at which the graph of an equation crosses the x-axis and the y-axis

interval an interval describes a set of numbers within which a solution falls

interval notation a mathematical statement that describes a solution set and uses parentheses or brackets to indicate where an interval begins and ends

linear equation an algebraic equation in which each term is either a constant or the product of a constant and the first power of a variable

linear inequality similar to a linear equation except that the solutions will include sets of numbers

midpoint formula a formula to find the point that divides a line segment into two parts of equal length

ordered pair a pair of numbers indicating horizontal displacement and vertical displacement from the origin; also known as a coordinate pair, (x, y)

origin the point where the two axes cross in the center of the plane, described by the ordered pair $(0, 0)$

perimeter in linear units, the perimeter formula is used to find the linear measurement, or outside length and width, around a two-dimensional regular object; for a rectangle: $P = 2L + 2W$

polynomial equation an equation containing a string of terms including numerical coefficients and variables raised to whole-number exponents

Pythagorean Theorem a theorem that states the relationship among the lengths of the sides of a right triangle, used to solve right triangle problems

quadrant one quarter of the coordinate plane, created when the axes divide the plane into four sections

quadratic equation an equation containing a second-degree polynomial; can be solved using multiple methods

quadratic formula a formula that will solve all quadratic equations

radical equation an equation containing at least one radical term where the variable is part of the radicand

rational equation an equation consisting of a fraction of polynomials

slope the change in y-values over the change in x-values

solution set the set of all solutions to an equation

square root property one of the methods used to solve a quadratic equation, in which the x^2 term is isolated so that the square root of both sides of the equation can be taken to solve for x

volume in cubic units, the volume measurement includes length, width, and depth: $V = LWH$

***x*-axis** the common name of the horizontal axis on a coordinate plane; a number line increasing from left to right

***x*-coordinate** the first coordinate of an ordered pair, representing the horizontal displacement and direction from the origin

***x*-intercept** the point where a graph intersects the *x*-axis; an ordered pair with a *y*-coordinate of zero

***y*-axis** the common name of the vertical axis on a coordinate plane; a number line increasing from bottom to top

***y*-coordinate** the second coordinate of an ordered pair, representing the vertical displacement and direction from the origin

***y*-intercept** a point where a graph intercepts the *y*-axis; an ordered pair with an *x*-coordinate of zero

zero-product property the property that formally states that multiplication by zero is zero, so that each factor of a quadratic equation can be set equal to zero to solve equations

Key Equations

quadratic formula	$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Distance	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Slope of a line	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Point-Slope equation of a line	$y - y_1 = m(x - x_1)$
Adding Complex Numbers	$(a + bi) + (c + di) = (a + c) + (b + d)i$
Subtracting Complex Numbers	$(a + bi) - (c + di) = (a - c) + (b - d)i$
Square Root Property	If $x^2 = k$ then $x = \pm\sqrt{k}$
Quadratic Formula	If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Rational Exponents	$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{m}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
Absolute Value	If $x \geq 0, \lvert x\rvert = x$; If $x < 0, \lvert x\rvert = -x$
Addition Property of Inequalities	If $a < b$ then $a + c < b + c$
Multiplication Property of Inequalities	If $a < b$ and $c > 0$, then $ac < bc$ If $a < b$ and $c < 0$, then $ac > bc$

Key Concepts

2.1 The Rectangular Coordinate Systems and Graphs

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the *x*-axis and displacement from the *y*-axis. See Example 1.
- An equation can be graphed in the plane by creating a table of values and plotting points. See Example 2.
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form *y*=____. See Example 3.
- Finding the *x*- and *y*-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See Example 4.
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See Example 5 and Example 6.
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the *x*-coordinates and the sum of the *y*-coordinates of the endpoints by 2. See Example 7 and Example 8.

2.2 Linear Equations in One Variable

- We can solve linear equations in one variable in the form $ax + b = 0$ using standard algebraic properties. See Example 1 and Example 2.
- A rational expression is a quotient of two polynomials. We use the LCD to clear the fractions from an equation. See Example 3 and Example 4.
- All solutions to a rational equation should be verified within the original equation to avoid an undefined term, or zero in the denominator. See Example 5 and Example 6 and Example 7.
- Given two points, we can find the slope of a line using the slope formula. See Example 8.
- We can identify the slope and *y*-intercept of an equation in slope-intercept form. See Example 9.
- We can find the equation of a line given the slope and a point. See Example 10.
- We can also find the equation of a line given two points. Find the slope and use the point-slope formula. See Example 11.
- The standard form of a line has no fractions. See Example 12.
- Horizontal lines have a slope of zero and are defined as $y = c$, where c is a constant.
- Vertical lines have an undefined slope (zero in the denominator), and are defined as $x = c$, where c is a constant. See Example 13.
- Parallel lines have the same slope and different *y*-intercepts. See Example 14 and Example 15.
- Perpendicular lines have slopes that are negative reciprocals of each other unless one is horizontal and the other is vertical. See Example 16.

2.3 Models and Applications

- A linear equation can be used to solve for an unknown in a number problem. See Example 1.
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities. See Example 2.
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d = rt$ formula. See Example 3.
- Many geometry problems are solved using the perimeter formula $P = 2L + 2W$, the area formula $A = LW$, or the volume formula $V = LWH$. See Example 4, Example 5, and Example 6.

2.4 Complex Numbers

- The square root of any negative number can be written as a multiple of i. See Example 1.
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis. See Example 2.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts. See Example 3.
- Complex numbers can be multiplied and divided.
  - To multiply complex numbers, distribute just as with polynomials. See Example 4 and Example 5.
  - To divide complex numbers, multiply both numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator. See Example 6 and Example 7.
- The powers of i are cyclic, repeating every fourth one. See Example 8.

2.5 Quadratic Equations

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of 1 or if the equation is a difference of squares. The zero-product property is then used to find solutions. See Example 1, Example 2, and Example 3.
- Many quadratic equations with a leading coefficient other than 1 can be solved by factoring using the grouping method. See Example 4 and Example 5.
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution. See Example 6 and Example 7.
- Completing the square is a method of solving quadratic equations when the equation cannot be factored. See Example 8.
- A highly dependable method for solving quadratic equations is the quadratic formula, based on the coefficients and the constant term in the equation. See Example 9 and Example 10.
- The discriminant is used to indicate the nature of the roots that the quadratic equation will yield: real or complex, rational or irrational, and how many of each. See Example 11.
- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems

and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation. See Example 12.

2.6 Other Types of Equations

- Rational exponents can be rewritten several ways depending on what is most convenient for the problem. To solve, both sides of the equation are raised to a power that will render the exponent on the variable equal to 1. See Example 1, Example 2, and Example 3.
- Factoring extends to higher-order polynomials when it involves factoring out the GCF or factoring by grouping. See Example 4 and Example 5.
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index. See Example 6 and Example 7.
- To solve absolute value equations, we need to write two equations, one for the positive value and one for the negative value. See Example 8.
- Equations in quadratic form are easy to spot, as the exponent on the first term is double the exponent on the second term and the third term is a constant. We may also see a binomial in place of the single variable. We use substitution to solve. See Example 9 and Example 10.
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form. See Example 11.

2.7 Linear Inequalities and Absolute Value Inequalities

- Interval notation is a method to indicate the solution set to an inequality. Highly applicable in calculus, it is a system of parentheses and brackets that indicate what numbers are included in a set and whether the endpoints are included as well. See Table 1 and Example 2**.**
- Solving inequalities is similar to solving equations. The same algebraic rules apply, except for one: multiplying or dividing by a negative number reverses the inequality. See Example 3**,** Example 4, Example 5, and Example 6**.**
- Compound inequalities often have three parts and can be rewritten as two independent inequalities. Solutions are given by boundary values, which are indicated as a beginning boundary or an ending boundary in the solutions to the two inequalities. See Example 7 and Example 8.
- Absolute value inequalities will produce two solution sets due to the nature of absolute value. We solve by writing two equations: one equal to a positive value and one equal to a negative value. See Example 9 and Example 10**.**
- Absolute value inequalities can also be solved by graphing. At least we can check the algebraic solutions by graphing, as we cannot depend on a visual for a precise solution. See Example 11.

Exercises

Review Exercises

The Rectangular Coordinate Systems and Graphs

For the following exercises, find the x-intercept and the y-intercept without graphing.

1. $4x-3y=12$

2. $2y-4=3x$

For the following exercises, solve for y in terms of x, putting the equation in slope–intercept form.

3. $5x=3y-12$

4. $2x-5y=7$

For the following exercises, find the distance between the two points.

5. $(-2,5)\,(4,-1)$

6. $(-12,-3)\,(-1,5)$

7. Find the distance between the two points $(-71{,}432)$ and $(511{,}218)$ using your calculator, and round your answer to the nearest thousandth.

For the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

8. $(-1,5)$ and $(4,6)$

9. $(-13,5)$ and $(17,18)$

For the following exercises, construct a table and graph the equation by plotting at least three points.

10. $y = \frac{1}{2}x + 4$

11. $4x - 3y = 6$

Linear Equations in One Variable

For the following exercises, solve for x.

12. $5x + 2 = 7x - 8$

13. $3(x + 2) - 10 = x + 4$

14. $7x - 3 = 5$

15. $12 - 5(x + 1) = 2x - 5$

16. $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$

For the following exercises, solve for x. State all x-values that are excluded from the solution set.

17. $\frac{x}{x^2-9} + \frac{4}{x+3} = \frac{3}{x^2-9}$
$x \neq 3, -3$

18. $\frac{1}{2} + \frac{2}{x} = \frac{3}{4}$

For the following exercises, find the equation of the line using the point-slope formula.

19. Passes through these two points: $(-2, 1), (4, 2)$.

20. Passes through the point $(-3, 4)$ and has a slope of $-\frac{1}{3}$.

21. Passes through the point $(-3, 4)$ and is parallel to the graph $y = \frac{2}{3}x + 5$.

22. Passes through these two points: $(5, 1), (5, 7)$.

Models and Applications

For the following exercises, write and solve an equation to answer each question.

23. The number of male fish in the tank is five more than three times the number of females. If the total number of fish is 73, how many of each sex are in the tank?

24. A landscaper has 72 ft. of fencing to put around a rectangular garden. If the length is 3 times the width, find the dimensions of the garden.

25. A truck rental is $25 plus $.30/mi. Find out how many miles Ken traveled if his bill was $50.20.

Complex Numbers

For the following exercises, use the quadratic equation to solve.

26. $x^2 - 5x + 9 = 0$

27. $2x^2 + 3x + 7 = 0$

For the following exercises, name the horizontal component and the vertical component.

28. $4 - 3i$

29. $-2 - i$

For the following exercises, perform the operations indicated.

30. $(9 - i) - (4 - 7i)$

31. $(2 + 3i) - (-5 - 8i)$

32. $2\sqrt{-75} + 3\sqrt{25}$

33. $\sqrt{-16} + 4\sqrt{-9}$

34. $-6i(i - 5)$

35. $(3 - 5i)^2$

36. $\sqrt{-4} \cdot \sqrt{-12}$

37. $\sqrt{-2}\left(\sqrt{-8} - \sqrt{5}\right)$

38. $\frac{2}{5-3i}$

39. $\frac{3+7i}{i}$

Quadratic Equations

For the following exercises, solve the quadratic equation by factoring.

40. $2x^2 - 7x - 4 = 0$

41. $3x^2 + 18x + 15 = 0$

42. $25x^2 - 9 = 0$

43. $7x^2 - 9x = 0$

For the following exercises, solve the quadratic equation by using the square-root property.

44. $x^2 = 49$

45. $(x-4)^2 = 36$

For the following exercises, solve the quadratic equation by completing the square.

46. $x^2 + 8x - 5 = 0$

47. $4x^2 + 2x - 1 = 0$

For the following exercises, solve the quadratic equation by using the quadratic formula. If the solutions are not real, state No real solution.

48. $2x^2 - 5x + 1 = 0$

49. $15x^2 - x - 2 = 0$

For the following exercises, solve the quadratic equation by the method of your choice.

50. $(x-2)^2 = 16$

51. $x^2 = 10x + 3$

Other Types of Equations

For the following exercises, solve the equations.

52. $x^{\frac{3}{2}} = 27$

53. $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} = 0$

54. $4x^3 + 8x^2 - 9x - 18 = 0$

55. $3x^5 - 6x^3 = 0$

56. $\sqrt{x+9} = x - 3$

57. $\sqrt{3x+7} + \sqrt{x+2} = 1$

58. $|3x - 7| = 5$

59. $|2x + 3| - 5 = 9$

Linear Inequalities and Absolute Value Inequalities

For the following exercises, solve the inequality. Write your final answer in interval notation.

60. $5x - 8 \leq 12$

61. $-2x + 5 > x - 7$

62. $\frac{x-1}{3} + \frac{x+2}{5} \leq \frac{3}{5}$

63. $|3x + 2| + 1 \leq 9$

64. $|5x - 1| > 14$

65. $|x - 3| < -4$

For the following exercises, solve the compound inequality. Write your answer in interval notation.

66. $-4 < 3x + 2 \leq 18$

67. $3y < 1 - 2y < 5 + y$

For the following exercises, graph as described.

68. Graph the absolute value function and graph the constant function. Observe the points of intersection and shade the *x*-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.
$|x + 3| \geq 5$

69. Graph both straight lines (left-hand side being y1 and right-hand side being y2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the *y*-values of the lines. See the interval where the inequality is true.
$x + 3 < 3x - 4$

Practice Test

1. Graph the following:
$2y = 3x + 4$.

2. Find the *x*- and *y*-intercepts for the following:
$2x - 5y = 6$

3. Find the *x*- and *y*-intercepts of this equation, and sketch the graph of the line using just the intercepts plotted.
$3x - 4y = 12$

4. Find the exact distance between $(5, -3)$ and $(-2, 8)$. Find the coordinates of the midpoint of the line segment joining the two points.

5. Write the interval notation for the set of numbers represented by $\{x | x \leq 9\}$.

6. Solve for x: $5x + 8 = 3x - 10$.

7. Solve for x: $3(2x - 5) - 3(x - 7) = 2x - 9$.

8. Solve for x: $\frac{x}{2} + 1 = \frac{4}{x}$

9. Solve for x: $\frac{5}{x+4} = 4 + \frac{3}{x-2}$.

10. The perimeter of a triangle is 30 in. The longest side is 2 less than 3 times the shortest side and the other side is 2 more than twice the shortest side. Find the length of each side.

11. Solve for x. Write the answer in simplest radical form. $\frac{x^2}{3} - x = -\frac{1}{2}$

12. Solve: $3x - 8 \leq 4$.

13. Solve: $|2x + 3| < 5$.

14. Solve: $|3x - 2| \geq 4$.

For the following exercises, find the equation of the line with the given information.

15. Passes through the points $(-4, 2)$ and $(5, -3)$.

16. Has an undefined slope and passes through the point $(4, 3)$.

17. Passes through the point $(2, 1)$ and is perpendicular to $y = -\frac{2}{5}x + 3$.

18. Add these complex numbers: $(3 - 2i) + (4 - i)$.

19. Simplify: $\sqrt{-4} + 3\sqrt{-16}$.

20. Multiply: $5i(5 - 3i)$.

21. Divide: $\frac{4-i}{2+3i}$.

22. Solve this quadratic equation and write the two complex roots in $a + bi$ form: $x^2 - 4x + 7 = 0$.

23. Solve: $(3x - 1)^2 - 1 = 24$.

24. Solve: $x^2 - 6x = 13$.

25. Solve: $4x^2 - 4x - 1 = 0$

26. Solve: $\sqrt{x - 7} = x - 7$

27. Solve: $2 + \sqrt{12 - 2x} = x$

28. Solve: $(x - 1)^{\frac{2}{3}} = 9$

For the following exercises, find the real solutions of each equation by factoring.

29. $2x^3 - x^2 - 8x + 4 = 0$

30. $(x + 5)^2 - 3(x + 5) - 4 = 0$