

1. (10 points) Let $A = \begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ 2 & -2 & -11 \end{bmatrix}$ and let $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Explain how the fact that the reduced row echelon form of $[A \mid \mathbf{w}]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

implies each of the following facts.

- (a) The columns $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ of A form a basis of $\mathbb{R}^3$. (Hint: to be a *basis*, $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must satisfy two properties. What are they? How do you know from the reduced row echelon form they hold?)

A basis of a vector space V is a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq V$ that

- (1) spans V (every element of V is a linear combination of elements of S);
- (2) is linearly independent (only way that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$ is if $c_1 = c_2 = \dots = c_n = 0$).

Since the RREF of A is I_3 it spans (since there are 3 pivots), and is linearly independent (since there is a pivot in every column).

- (b) The vector $\mathbf{w}$ can be written as a unique linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Use the reduced row echelon form of $[A \mid \mathbf{w}]$ to find the particular linear combination and explain why/how we know it is unique.

A solution to $A\vec{x} = \vec{w}$ is a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

for which $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{w}$. Solutions to this match up with solutions to $I_3\vec{x} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ since

row operations preserve the set of solutions. There is only one solution to $I_3\vec{x} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, i.e. $\vec{x} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$.

$$\begin{aligned} \text{We have: } 2\mathbf{v}_1 - 5\mathbf{v}_2 + \mathbf{v}_3 &= 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 5 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \\ -11 \end{bmatrix} \\ &= \begin{bmatrix} 2 & +5 & -6 \\ 0 & -5 & +4 \\ 4 & +10 & -11 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \mathbf{w} \quad \checkmark \end{aligned}$$

2. Suppose that a computer manufacturer makes two models of computer, Model A and Model B. It is estimated that the profit on Model A is \$4 per unit, and the profit on Model B is \$10 per unit. Suppose that the time (in hours) required to wire, assemble and test each computer, and the total available labor hours is as follows:

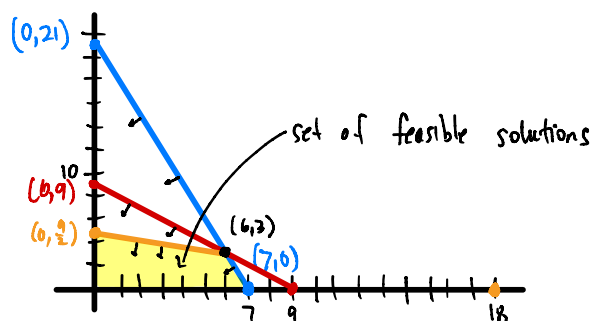
	Model A	Model B	Available labor
Wiring	1	4	18
Assembly	3	1	21
Testing	1	1	9

Note that this scenario can be modeled with the following linear programming problem.

Maximize $z = 4x_1 + 10x_2$ subject to

$$\left. \begin{array}{l} x_1 + 4x_2 \leq 18, \\ 3x_1 + x_2 \leq 21, \\ x_1 + x_2 \leq 9, \\ x_1, x_2 \geq 0. \end{array} \right\} \begin{array}{l} (0,0) \text{ satisfies} \\ \text{each of these} \\ \text{so half plane includes origin} \\ \text{for each.} \end{array} \quad \text{Quadrant I}$$

- (a) (15 points) Use the geometric method to find a solution to this problem. What are the values of the slack variables for your solution?



$$L_1: (0, \frac{9}{2}) \quad (18, 0)$$

$$L_2: (0, 21) \quad (7, 0)$$

$$L_3: (0, 9) \quad (9, 0)$$

The point of intersection of L_1, L_2, L_3 corresponds to solution to $\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 21 \\ 9 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 4 & 18 \\ 3 & 1 & 21 \\ 1 & 1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 18 \\ 0 & 3 & 9 \\ 0 & -2 & -6 \end{bmatrix} \xrightarrow{\substack{R_2 - R_3 \\ R_3 - 3R_2}} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\substack{R_1 - \frac{1}{3}R_2 \\ -\frac{1}{2}R_3}} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \cdot n_2} \Rightarrow \begin{array}{l} x=6 \\ y=3 \end{array}$$

Extreme Point	z-value
(0,0)	0
(7,0)	28
(6,3)	$24 + 30 = 54$
(0,9)	45

By extreme point theorem, since the set of feasible solutions is bounded, optimal value occurs at extreme pt.

⇒ Optimal value is $z = 54$ at $(x,y) = (6,3)$.

$$(x_1, x_2) = (6, 3)$$

$$z = 54$$

$$\text{slack variables} = (0, 0, 0)$$

Slack variables

$$x_3 = 18 - (6 + 4 \cdot 3) = 0$$

$$x_4 = 21 - (3 \cdot 6 + 3) = 0$$

$$x_5 = 9 - (6 + 3) = 0$$

- (b) (5 points) Convert the problem to canonical form. (Be sure to write down all elements of the problem—not just the part that may change from that given above.)

$$\text{Maximize } z = 4x_1 + 10x_2 \text{ subject to}$$

$$x_1 + 4x_2 + x_3 = 18$$

$$3x_1 + x_2 + x_4 = 21$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

- (c) (20 points) Solve the problem using the simplex method.

		4	10	0	0	0	
		x_1	x_2	x_3	x_4	x_5	$\rightarrow X_B$
0	x_3	1	4	1	0	0	18
0	x_4	3	1	0	1	0	21
0	x_5	1	1	0	0	1	9
		-4	-10	0	0	0	0
		4	10	0	0	0	$\rightarrow X_B$
10	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	$\frac{9}{2}$
0	x_4	$\frac{11}{4}$	0	$-\frac{1}{4}$	1	0	$\frac{33}{2}$
0	x_5	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	1	$\frac{9}{2}$
		$-\frac{3}{2}$	0	$\frac{5}{2}$	0	0	45
		4	10	0	0	0	$\rightarrow X_B$
10	x_2	0	1	$\frac{3}{11}$	$-\frac{1}{11}$	0	3
4	x_1	1	0	$-\frac{1}{11}$	$\frac{4}{11}$	0	6
0	x_5	0	0	$-\frac{2}{11}$	$-\frac{3}{11}$	1	0
		0	0	$\frac{26}{11}$	$\frac{6}{11}$	0	54

θ -ratios

$$\frac{18}{4} = \frac{9}{2} \leftarrow \text{minimal positive}$$

$$\frac{21}{1}$$

$$\frac{9}{1}$$

θ -ratios

$$\frac{9}{2} \cdot \frac{4}{1} = 18$$

$$\frac{33}{2} \cdot \frac{4}{11} = 6$$

$$\frac{9}{2} \cdot \frac{4}{3} = 6$$

$$x_0 = (6, 3, 0, 0, 0)$$

$$z = 54$$

- (d) (20 points) Recall that if we write the constraints (in canonical form) as $A\mathbf{x} = \mathbf{b}$, the algebraic method of solving the problem requires one to check that columns A_i, A_j, A_k (or, equivalently the variables x_i, x_j, x_k) are *basic feasible*.

i. How many choices of $B = (i, j, k)$ are there that need to be considered?

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2} = 10 \quad \text{Number of choices} = \boxed{\binom{5}{3} = 10}.$$

ii. What properties must be satisfied by A_i, A_j, A_k in order to be basic? basic feasible?

Basic: A_i, A_j, A_k must be linearly independent

Basic Feasible: Basic + $x_i A_i + x_j A_j + x_k A_k = \mathbf{b}$
has $x_i, x_j, x_k \geq 0$.

iii. Determine (and justify) whether A_2, A_4, A_5 is basic and/or basic feasible. If so, find the corresponding basic (or basic feasible) solution.

$$[A_2 \ A_4 \ A_5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 21 \\ 9 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 4 & 0 & 0 & 18 \\ 1 & 1 & 0 & 21 \\ 1 & 0 & 1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{9}{2} \\ 1 & 1 & 0 & 21 \\ 1 & 0 & 1 & 9 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 0 & 0 & 4.5 \\ 0 & 1 & 0 & 16.5 \\ 0 & 0 & 1 & 4.5 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\Rightarrow \text{Basic and feasible} \quad \vec{x} = \begin{bmatrix} 9/2 \\ 33/2 \\ 9/2 \end{bmatrix}$$

iv. Determine (and justify) whether A_2, A_3, A_4 is basic and/or basic feasible. If so, find the corresponding basic (or basic feasible) solution.

$$\begin{bmatrix} 4 & 1 & 0 & 18 \\ 1 & 0 & 1 & 21 \\ 1 & 0 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 12 \\ 1 & 0 & 0 & 9 \end{bmatrix} \begin{matrix} R_1 - 4R_3 \\ R_2 - R_3 \\ \end{matrix}$$

$\Rightarrow$ Basic but not feasible.

3. (12 points) Convex sets in $\mathbb{R}^n$.

(a) What is a *hyperplane* in $\mathbb{R}^n$?

A hyperplane is the set of solutions to a single linear equation in n -variables: $\{\vec{x} \in \mathbb{R}^n \mid \vec{c}^T \vec{x} = k\}$ for some $\vec{c} \in \mathbb{R}^n$, $k \in \mathbb{R}$.

(b) What is a *half-space* in $\mathbb{R}^n$?

$\{\vec{x} \in \mathbb{R}^n \mid \vec{c}^T \vec{x} \geq k\}$ for some $\vec{c} \in \mathbb{R}^n$, $k \in \mathbb{R}$.

(c) What does it mean to say that a set $S \subseteq \mathbb{R}^n$ is convex?

If $\vec{x}_1, \vec{x}_2 \in S$ then the entire line segment connecting $\vec{x}_1$ and $\vec{x}_2$ is in S : $\{\vec{x}_1(1-t) + \vec{x}_2 t \mid 0 \leq t \leq 1\} \subset S$.

(d) Prove that any half-space in $\mathbb{R}^n$ is convex.

Suppose that $S = \{\vec{x} \in \mathbb{R}^n \mid \vec{c}^T \vec{x} \geq k\}$. Suppose $\vec{x}_1, \vec{x}_2 \in S$.

Need to show that $\vec{x}_1(1-t) + \vec{x}_2 t \in S$ for every $t \in [0, 1]$.

To do so multiply by $\vec{c}^T$:

$$\begin{aligned} \vec{c}^T(\vec{x}_1(1-t) + \vec{x}_2 t) &= \underbrace{\vec{c}^T \vec{x}_1}_{\geq k} (1-t) + \underbrace{\vec{c}^T \vec{x}_2}_{\geq k} t && \text{(since } \vec{x}_1, \vec{x}_2 \in S) \\ &\geq k(1-t) + kt && \text{(since } 1-t, t \geq 0) \\ &= k \end{aligned}$$

Therefore, $\vec{x}_1(1-t) + \vec{x}_2 t \in S$, and thus S is convex.

4. (18 points) Consider a linear programming problem with constraints $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 4 & 1 & 2 & 1 \\ -1 & 1 & 0 & 2 & 1 \\ -2 & 3 & 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Determine (with adequate justification) which of the following are basic and/or basic feasible solutions.

- (a) $(1, 0, 1, 1, 0)$
 (b) $(0, 2, -5, 0, -1)$
 (c) $(0, 0, 1, 0, 1)$

$$[A_1 \ A_3 \ A_4] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow$ Neither basic nor basic feasible.

$$[A_2 \ A_3 \ A_5] \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \checkmark$$

but $\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ does not have all non-negative entries, and $A_5 = A_2 - 3A_3$

so this solution is not basic (nor feasible)

$$[A_3 \ A_4 \ A_5] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \checkmark$$

$\uparrow$
 these
 are
 linearly independent
 since it has 3 pivots

$\Rightarrow$ basic &
 basic feasible.