CS 323: Numerical Analysis and Computing

Midterm #1

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Instructions: This is a **closed notes** exam. You are **not allowed** to consult any textbook, your class notes, homeworks, or any of the handouts from us. You are also **not permitted** to use laptop computers, cell phones, tablets, or any other hand-held electronic devices.

Name	
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Part #1	
Part #2	
Part #3	
Part #4	
Part #5	
TOTAL	

1. [24% = 4 questions × 6% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). No justification is required for your answer(s).

   (a) What is the best reason one might want to use the Secant method instead of Newton's method (for solving $f(x) = 0$)?

   (Circle the ONE most correct answer.)

   i. It is always guaranteed to converge, while Newton's method is not.

   ii. The Secant method can be used without having a formula for $f'(x)$.

   iii. Newton's method exhibits slower convergence than the Secant method.

   (b) Which of these statements about the Bisection method are true?

   (Circle the TWO most correct answers.)

   i. In order to use the Bisection method, we need to have a continuous function $f(x)$, with continuous derivative $f'(x)$.

   ii. The order of convergence for the Bisection method is linear.

   iii. We can use the Bisection method without any knowledge of the derivative $f'(x)$.

   (c) What happens if we use Newton's method to solve $f(x) = 0$, and the derivative $f'(x)$ happens to be zero at the exact location of the solution?

   (Circle the ONE most correct answer.)

   i. Newton's method should be avoided; the Secant method would be more robust in this case.

   ii. We may still be able to use Newton's method, if we have the explicit formula for $f(x)$. However, the order of convergence would degrade to linear.

   iii. Newton's method would still have quadratic convergence, but we would need to start with an initial guess that is very close to the solution.

   (d) What would happen if we try to use an n-degree polynomial to interpolate n data points?

   (Circle the ONE most correct answer.)

   i. This is the normal case. A unique polynomial can be determined.

   ii. Such an interpolant generally exists, but it is not unique.

   iii. Such an interpolant will generally not exist.

2. [24% = 3 questions × 8% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 2-3 sentences.

   (a) State (a) one advantage of the Vandermonde matrix method over Lagrange interpolation, and (b) one advantage of Newton interpolation over Lagrange interpolation.

   Answer: A polynomial constructed with the Vandermonde matrix method can be subsequently evaluated in $O(n)$ time (as opposed to $O(n^2)$ for Lagrange interpolation). Newton interpolation also allows for $O(n)$ evaluation cost.

   (b) Assuming the point sampling is fixed, why would we prefer using piecewise polynomial interpolation rather than a single global polynomial interpolant?

   Answer: High degree polynomials tend to be highly oscillatory, so it is preferable to use piecewise interpolation with low degree polynomials.

   (c) Give *two* reasons why the triangular matrix method for Newton interpolation might be preferable over the Vandermonde matrix method.

   Answer: The triangular matrix method for Newton interpolation produces a lower triangular matrix that can be solved in time $O(n^2)$ (as opposed to $O(n^3)$ for the Vandermonde matrix). The triangular matrix is also better behaved compared to the Vandermonde matrix which starts becoming close to singular as the degree of the polynomial interpolant increases.

3. [16%] Assume that the fixed point iteration methods $x_{k+1} = g(x_k)$ and $x_{k+1} = h(x_k)$ are *both* guaranteed to converge in the same interval, and their order of convergence is quadratic. Would the fixed point iteration

$$x_{k+1} = \frac{g(x_k) + h(x_k)}{2}$$

also exhibit quadratic convergence or not? Explain.

Answer: The new fixed point iteration is written as $x_{k+1} = q(x_k)$, where $q(x) = (g(x) + h(x))/2$. If the first two fixed point iterations have quadratic convergence, we have that $g'(a) = h'(a) = 0$ at the solution $x = a$. Consequently, $q'(a) = 0$ as well, and the corresponding fixed point iteration will converge quadratically also.

4. [16%] Using Newton interpolation, find a cubic polynomial that interpolates the four points:

$$(-2,-5), \quad (-1,-6), \quad (1,-8), \quad (2,3)$$

Answer:

x_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
-2	-5			
-1	-6	-1		
1	-8	-1	0	
2	3	0	4	1

Reading the coefficients of the Newton polynomials from the diagonal of this table, the final interpolant becomes

$$\begin{aligned} p(x) &= -5 \cdot 1 + (-1) \cdot (x+2) + 0 \cdot (x+2)(x+1) + 1 \cdot (x+2)(x+1)(x-1) \\ &= x^3 + 2x^2 - 2x - 9 \end{aligned}$$

5. [20%] Consider the nonlinear equation $f(x) = e^{3x} - 2 = 0$. Prove that Newton's method will always converge if started from a *positive* initial guess, and that convergence will be quadratic.

   Hint: If you write Newton's method in the form $x_{k+1} = g(x_k)$, you can prove the first part of the question by showing that $g(x)$ is a contraction for $x > 0$, and the second part by showing that $g'(a) = 0$, where a is the solution.

   Answer: The iteration function for Newton's method is:

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{e^{3x} - 2}{3e^{3x}} = x - \frac{1}{3} + \frac{2}{3}e^{-3x}$$

   and its derivative is

$$g'(x) = 1 - 2e^{-3x}$$

   The condition $|g'(x)| < 1$ is equivalently written as

$$\begin{aligned} |1 - 2e^{-3x}| &< 1 \\ \Leftrightarrow -1 < 1 - 2e^{-3x} &< 1 \\ \Leftrightarrow -2 < -2e^{-3x} &< 0 \\ \Leftrightarrow 1 > e^{-3x} &> 0 \end{aligned}$$

   which is true for all $x > 0$, indicating that g is a contraction in this case.
   The solution a satisfies $f(a) = 0 \Rightarrow e^{3a} - 2 = 0 \Rightarrow e^{3a} = 2$. Thus,

$$g'(a) = 1 - 2e^{-3a} = 1 - \frac{2}{e^{3a}} = 1 - \frac{2}{2} = 0$$

   and so it follows from Taylor's theorem that convergence is quadratic.