Math 354
Midterm Exam 1
February 25, 2026

Name: _____ netID: _____ Section: _____

Instructions

- This exam consists of 8 pages, including this one. If pages are missing, let the examiner(s) know right away.
- **Do not unstaple pages**.
- Show all relevant work. A solution without appropriate justification WILL NOT receive full credit.
- Write **clearly and legibly**. Points *will* be taken off if the grader cannot read the answer or understand the logical sequence. Your exam will be scanned, so it is important to use a writing implement which is sufficiently dark.
- You may use the back side of any sheet for scratch work. **This work will *not* be graded**.
- Answer each question on its own page. If you need more space, you may use the last page which is marked "EXTRA SHEET OF PAPER". If you do this be sure to:
  - Indicate in the space below the question that you are continuing the problem on the extra sheet (e.g., in the space designated for problem #1, write "problem #1 continues on the extra sheet")
  - Indicate on the extra sheet which problem you are working on (e.g., on the extra sheet, write "#1 continued")

Page:	2	3	4	5	6	7	Total
Points:	10	15	25	20	12	18	100
Score:							

1. (10 points) Let $A = \begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ 2 & -2 & -11 \end{bmatrix}$ and let $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Explain how the fact that the reduced row echelon form of $\left[\begin{array}{c|c} A & \mathbf{w} \end{array}\right]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

implies each of the following facts.

(a) The columns $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ of A form a basis of $\mathbb{R}^3$. (Hint: to be a *basis*, $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must satisfy two properties. What are they? How do you know from the reduced row echelon form they hold?)

(b) The vector $\mathbf{w}$ can be written as a unique linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Use the reduced row echelon form of $\left[\begin{array}{c|c} A & \mathbf{w} \end{array}\right]$ to find the particular linear combination and explain why/how we know it is unique.

2. Suppose that a computer manufacturer makes two models of computer, Model A and Model B. It is estimated that the profit on Model A is \$4 per unit, and the profit on Model B is \$10 per unit. Suppose that the time (in hours) required to wire, assemble and test each computer, and the total available labor hours is as follows:

	Model A	Model B	Available labor
Wiring	1	4	18
Assembly	3	1	21
Testing	1	1	9

Note that this scenario can be modeled with the following linear programming problem.

Maximize $z = 4x_1 + 10x_2$ subject to

$$\begin{aligned} x_1 + 4x_2 &\leq 18, \\ 3x_1 + x_2 &\leq 21, \\ x_1 + x_2 &\leq 9, \\ x_1, x_2 &\geq 0. \end{aligned}$$

(a) (15 points) Use the geometric method to find a solution to this problem. What are the values of the slack variables for your solution?

$(x_1, x_2) =$ $\quad$ $z =$ $\quad$ slack variables = .

(b) (5 points) Convert the problem to canonical form. (Be sure to write down all elements of the problem—not just the part that may change from that given above.)

(c) (20 points) Solve the problem using the simplex method.

$\mathbf{x}_0 =$ $\qquad$ $z =$.

(d) (20 points) Recall that if we write the constraints (in canonical form) as $A\mathbf{x} = \mathbf{b}$, the algebraic method of solving the problem requires one to check that columns A_i, A_j, A_k (or, equivalently the variables x_i, x_j, x_k) are *basic feasible*.

i. How many choices of $B = (i, j, k)$ are there that need to be considered?

Number of choices = $\boxed{}$.

ii. What properties must be satisfied by A_i, A_j, A_k in order to be basic? basic feasible?

iii. Determine (and justify) whether A_2, A_4, A_5 is basic and/or basic feasible. If so, find the corresponding basic (or basic feasible) solution.

iv. Determine (and justify) whether A_2, A_3, A_4 is basic and/or basic feasible. If so, find the corresponding basic (or basic feasible) solution.

3. (12 points) Convex sets in $\mathbb{R}^n$.

   (a) What is a *hyperplane* in $\mathbb{R}^n$?

   (b) What is a *half-space* in $\mathbb{R}^n$?

   (c) What does it mean to say that a set $S \subseteq \mathbb{R}^n$ is convex?

   (d) Prove that any half-space in $\mathbb{R}^n$ is convex.

4. (18 points) Consider a linear programming problem with constraints $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 4 & 1 & 2 & 1 \\ -1 & 1 & 0 & 2 & 1 \\ -2 & 3 & 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Determine (with adequate justification) which of the following are basic and/or basic feasible solutions.

(a) $(1, 0, 1, 1, 0)$

(b) $(0, 2, -5, 0, -1)$

(c) $(0, 0, 1, 0, 1)$

[EXTRA PAGE OF PAPER - DO NOT DETACH]