

Consider the linear programming problem to maximize $z = 4x_1 + 2x_2 + 7x_3$ subject to

$$2x_1 - x_2 + 4x_3 \leq 18$$

$$4x_1 + 2x_2 + 5x_3 \leq 10$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

1. Transform this problem into one in canonical form: Maximize $z = 4x_1 + 2x_2 + 7x_3$ subject to $Ax = b$. (Hint: A will be a 2×5 matrix.)

$$A = \begin{bmatrix} 2 & -1 & 4 & 1 & 0 \\ 4 & 2 & 5 & 0 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$$

2. Determine which subsets $I \subseteq \{1, 2, 3, 4, 5\}$ have the property that the columns A_i for $i \in I$ are linearly independent. For each one, find the corresponding basic solution. Is it feasible? (Use the back side of this sheet, as necessary.)

I	basic variables	columns of A	linearly independent?	basic solution	feasible?
$\{1, 2\}$	x_1, x_2	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$	Yes	$(\frac{23}{4}, -\frac{13}{2}, 0, 0, 0)$	No
$\{1, 3\}$	x_1, x_3	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}$	Yes	$(-\frac{28}{3}, 0, \frac{24}{3}, 0, 0)$	No
$\{1, 4\}$	x_1, x_4	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	Yes	$(\frac{5}{2}, 0, 0, 13, 0)$	Yes
$\{1, 5\}$	x_1, x_5	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	Yes	$(9, 0, 0, 0, -26)$	No
$\{2, 3\}$	x_2, x_3	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}$	Yes	$(0, -\frac{38}{13}, \frac{46}{13}, 0, 0)$	No
$\{2, 4\}$	x_2, x_4	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	Yes	$(0, 5, 0, 23, 0)$	Yes
$\{2, 5\}$	x_2, x_5	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	Yes	$(0, -18, 0, 0, 46)$	No
$\{3, 4\}$	x_3, x_4	$\begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	Yes	$(0, 0, 2, 10, 0)$	Yes
$\{3, 5\}$	x_3, x_5	$\begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	Yes	$(0, 0, \frac{9}{2}, 0, -\frac{25}{2})$	No
$\{4, 5\}$	x_4, x_5	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	Yes	$(0, 0, 0, 18, 10)$	Yes

Computation of
solution on
back side

$$(0, -\frac{50}{13}, \frac{46}{13}, 0, 0)$$

3. Find the optimal solution.

The above work shows that there are 5 extreme points. The optimal solution is the largest corresponding value of z . By work on back side, we see that this is $z = 14$

$$I \quad [A_i, A_{i_1} \vdots \vec{b}]$$

$$\{1,2\} \quad \begin{bmatrix} 2 & -1 & 18 \\ 4 & 2 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 18 \\ 0 & 4 & -26 \end{bmatrix} R_2 - 2R_1 \rightarrow \begin{bmatrix} 2 & -1 & 18 \\ 0 & 1 & -13/2 \end{bmatrix} R_2/4 \rightarrow \begin{bmatrix} 2 & 0 & 23/2 \\ 0 & 1 & -13/2 \end{bmatrix}$$

$$\Rightarrow x_1 = \frac{23}{4}, \quad x_2 = -\frac{13}{2}$$

$$\{1,3\} \quad \begin{bmatrix} 2 & 4 & 18 \\ 4 & 5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 18 \\ 0 & -3 & -26 \end{bmatrix} R_2 - 2R_1 \rightarrow \begin{bmatrix} 2 & 4 & 18 \\ 0 & 1 & 26/3 \end{bmatrix} R_1/3 \rightarrow \begin{bmatrix} 2 & 0 & 48-104/3 \\ 0 & 1 & 26/3 \end{bmatrix} R_1 - 4R_2 \quad \begin{array}{l} 94 \\ -18 \\ 56 \end{array}$$

$$\Rightarrow x_1 = -\frac{28}{3}, \quad x_3 = \frac{26}{3}$$

$$\{1,4\} \quad \begin{bmatrix} 2 & 1 & 18 \\ 4 & 0 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 18 \\ 0 & -2 & -26 \end{bmatrix} R_1 - \frac{1}{2}R_2 \Rightarrow x_1 = \frac{5}{2}, \quad x_4 = 13$$

$$\{1,5\} \quad \begin{bmatrix} 2 & 0 & 18 \\ 4 & 1 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 18 \\ 0 & 1 & -26 \end{bmatrix} \Rightarrow x_1 = 9, \quad x_5 = -26$$

$$\{2,3\} \quad \begin{bmatrix} -1 & 4 & 18 \\ 2 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 4 & 18 \\ 0 & 13 & 46 \end{bmatrix} R_1 + 2R_2 \rightarrow \begin{bmatrix} -1 & 4 & 18 \\ 0 & 1 & 46/13 \end{bmatrix} \xrightarrow{R_1 \cdot 13} \begin{bmatrix} -13 & 52 & 234 \\ 0 & 1 & 46/13 \end{bmatrix} \xrightarrow{R_1 + 13R_2} \begin{bmatrix} -13 & 52 & 234 \\ 0 & 1 & 46/13 \end{bmatrix} \Rightarrow x_3 = \frac{46}{13}, \quad x_2 = -18 + \frac{4 \cdot 46}{13} = \frac{-18 \cdot 13 + 184}{13} = \frac{-50}{13}$$

$$\{2,4\} \quad \begin{bmatrix} -1 & 1 & 18 \\ 2 & 0 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 18 \\ 0 & 2 & 46 \end{bmatrix} R_1 + 2R_2 \rightarrow \begin{bmatrix} -1 & 1 & 18 \\ 0 & 1 & 23 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} -1 & 0 & -5 \\ 0 & 1 & 23 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 23 \end{bmatrix} \quad x_2 = 5, \quad x_4 = 23$$

$$\{2,5\} \quad \begin{bmatrix} -1 & 0 & 18 \\ 2 & 1 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & 18 \\ 0 & 1 & 46 \end{bmatrix} \Rightarrow x_2 = -18, \quad x_5 = 46$$

$$\{3,4\} \quad \begin{bmatrix} 4 & 1 & 18 \\ 5 & 0 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 1 & 18 \\ 1 & -1 & -8 \end{bmatrix} R_2 - R_1 \rightarrow \begin{bmatrix} 0 & 5 & 50 \\ 1 & -1 & -8 \end{bmatrix} R_1 - 4R_2 \rightarrow \begin{bmatrix} 0 & 1 & 10 \\ 1 & -1 & -8 \end{bmatrix} \xrightarrow{\frac{1}{5}R_1} \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -8 \end{bmatrix} \Rightarrow x_4 = 2, \quad x_3 = 10$$

$$\{3,5\} \quad \begin{bmatrix} 4 & 0 & 18 \\ 5 & 1 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 0 & 18 \\ 1 & 1 & -8 \end{bmatrix} R_1 - R_2 \rightarrow \begin{bmatrix} 0 & -4 & 50 \\ 1 & 1 & -8 \end{bmatrix} R_1 - 4R_2 \rightarrow \begin{bmatrix} 0 & 1 & -25/2 \\ 1 & 1 & -8 \end{bmatrix} \Rightarrow x_5 = -\frac{25}{2}, \quad x_3 = -8 + \frac{25}{2} = \frac{9}{2}$$

$$\{4,5\} \quad \begin{bmatrix} 1 & 0 & 18 \\ 0 & 1 & 10 \end{bmatrix} \quad x_4 = 18, \quad x_5 = 10$$

$\vec{x}$	$z = 4x_1 + 2x_2 + 7x_3$
$(\frac{5}{2}, 0, 0, 13, 0)$	10
$(0, 5, 0, 23, 0)$	10
$(0, 0, 2, 10, 0)$	14 ← maximal value
$(0, 0, 0, 18, 10)$	0